Egalitarianism Under Earmark Constraints *

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Abstract

We consider a model in which a homogeneous commodity (the resource) is shared by several agents with single-peaked preferences, but the resource is coming from any number of different suppliers, under arbitrary bilateral feasibility constraints: each supplier can only deliver to a certain subset of agents. Examples include balancing the workload of machines, sharing earmarked funds between different projects, and assigning students to schools under geographic constraints.

Unlike in the one supplier model (Sprumont, 1991), that we generalize, in a Pareto optimal allocation agents who get more than their peak typically coexist with agents who get less. A variant of the Gallai-Edmonds decomposition (see Ore, 1962) identifies these two subsets of agents, that we call respectively the over-demanded and the under-demanded side of the market. Like in the one supplier model, there is a Lorenz dominant Pareto optimal allocation. We call it the *egalitarian* solution, and characterize it by the combination of strategyproofness (truthful revelation of peaks) and a variant of equal treatment of equals. the proof relies on submodular optimization techniques as in Dutta and Ray (1989).

Keywords: Bipartite graph, Lorenz dominant solution, Strategy-proofness, Equal treatment of equals, Single-peaked preferences.

JEL codes: C72, D63, D61, C78, D71.

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*The authors warmly thank Anne Bogomolnaia and Jay Sethuraman for stimulating discussions.

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1 Introduction

Egalitarianism is the central principle of fair division that may conflict with a number of incentives, feasibility or efficiency constraints. Maximizing the leximin ordering over profiles of relevant characteristics (a. k. a. the Rawlsian approach) is the most common implementation of egalitarianism under constraints. It is however a controversial method, that recommends to take arbitrarily large amounts of resources from the "rich" if this allows to raise by even a tiny amount the lot of the "poor". The only case where egalitarianism eschews this devastating critique is when we can find a Lorenz dominant distribution of welfare, or resources: at the Lorenz dominant outcome, we simultaneously maximize the share of the k poorest individuals, for *any number* k of agents[1]. Unlike the leximin ordering that always reaches a unique maximum in any closed convex set, a Lorenz dominant outcome may not exist in such sets. We know in fact very few fair division models admitting Lorenz dominating solutions over a reasonably rich domain of problems. The two main exceptions follow.

Dutta and Ray (1989) observed that the core of a supermodular (convex) cooperative game is one very general instance where a Lorenz dominant solution exists; this solution has been known after their work as the *egalitarian* selection in the core. The second model is the fair division of a single commodity under single-peaked preferences and no free disposal (Benassy, 1982, Sprumont, 1991). The *uniform solution* selects for each agent either his peak, or a common share in such a way that the resource is fully distributed. Although the original motivation of the uniform solution was its incentive properties (Benassy, 1982), its most compelling fairness property, and its shortest definition, is to be Lorenz dominant among all Pareto optimal allocations of the resource (De Frutos and Massó, 1995).

We study a considerable generalization of the Sprumont model, where a homogeneous commodity (the resource) is still shared by several agents with single-peaked preferences, but the resource is coming from any number of different suppliers, under arbitrary bilateral feasibility constraints: each supplier can only deliver to a certain subset of agents. Examples where such constraints are critical include:

- Balancing the workloads of several machines, when each machine can only process certain jobs, but the processing speed is uniform. Assigning customers to service persons when language constraints limits the set of customers each service person can handle.
- Sharing earmarked resources when the earmarking is not one-to-one[2], for instance dividing funds between different research projects, when the foundations, agencies, or private donors put overlapping constraints on the use of their gift. For instance one donor funds projects relevant to global

[1] Reference on the Lorenz optimum: Sen's 1970; Moulin 1988; Foster and Sen 2003.

[2] As opposed to the usual meaning of an earmark as "*a provision in Congressional legislation that allocates a specified amount of money for a specific project, program, or organization*" (Merriam-Webster dictionary).

warming, another donor looks for projects with a Latin American component, a third one for those involving minorities, etc..

- Assigning students to schools when each school can only handle a certain subset of students –e.g., those coming from certain neighborhoods–, and.these subsets overlap.

Assuming that each recipient of the resources want to maximize her share if the resource is a "good" (money), or minimize it if it is a "bad" (workload) is a reasonable first approximation. But in most concrete examples the situation is more nuanced. Under the widespread bureaucratic constraint that funds must be spent in a given calendar year, and the belief that returning funds has a negative impact on future funding, a project manager does not want her budget to be too large, lest it becomes difficult to find justifiable ways to spend it. If the workload of a worker is too small, his machine or his job may soon be deemed redundant, so his most preferred workload is not zero. Similarly a school principal has in mind an ideal amount of students she would like to handle, not too large so that classes will not be crowded, not too small lest some of his faculty or staff becomes idle. And so on.

Single-peaked preferences provide a rich model of the agents' goals, from always increasing (more commodity is always better) to always decreasing (less is always better), and much in between. But the target share of an agent depends upon many subjective, privately known factors, therefore any division rule, fair or otherwise, to allocate the resources must worry about the agents' incentives to truthfully reveal their true target. We take *strategyproofness* (truthful report is a dominant strategy) as our incentive compatibility design constraint. We identify a canonical division rule that simultaneously aligns incentives with efficiency (is strategyproof and selects a Pareto optimal allocation), and is egalitarian-fair, in the sense that it selects the Lorenz dominant Pareto optimal allocation.

A simple blood division example will help develop intuition for our *egalitarian* solution. A blood bank must divide the (objective) blood needs of a group of patients between a set of donors; patients and donors are partitioned by blood type and transfusions must respect the familiar compatibility constraints: (i) type O are universal donors (ii) type AB are universal receivers (iii) type A can also give to A , (iii) type B can also give to B (iv) type AB can only give to AB.

Figure 1 shows on the right side the quantity of blood needed by each group of receivers of the same type. On the left side a node is a group of donors of a given type. The total demand of 40 units must be served, and we assume first that the bank wishes to share the burden equally among the blood types. Taking 10 units from each group of donors is clearly not feasible. The most egalitarian division of the burden compatible with Figure 1 is: $AB:6$, $B:8$, $A:13$, $O:13$. Type AB donors cannot contribute more than 6 units toward the patients' demand. Given AB's share, type B donors give as much as type

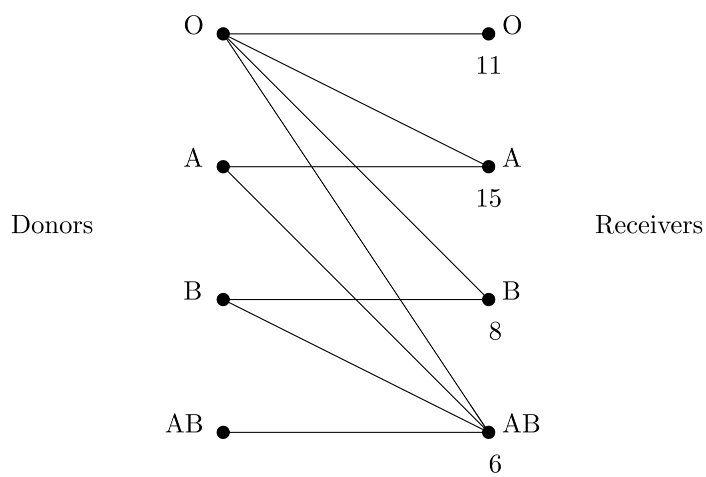

Figure 1: The donor-receiver example I

AB and B receivers can still accept. Finally, types A and O donors supply equal amounts to cover the rest of the demand.

A more refined version of the model takes into account the fact that the distribution of blood types is not homogeneous[3]. Ideally the blood bank wishes to spend the blood it receives from various groups of donors in proportion to the representation of these groups in the population. We assume that those proportions are $AB : 12.5\%$, $B : 25\%$, $A : 37.5\%$, $O : 25\%$, so the ideal distribution of the demand in the example of Figure 1 is as follows

Consider the egalitarian allocation $AB : 6$, $B : 8$, $A : 13$, $O : 13$ identified above. It uses more type AB blood than the target of 5, and less of B blood than the target of 10. Similarly type A is giving more than its target 10, and type O is giving less than 15. Within the compatibility constraints, we can rearrange the shares of type AB and type B as $AB : 5$, $B : 9$, which will bring the contribution of both types closer to their target; similarly we can modify the shares of types A and O as $A : 15$, $O : 11$. When each node on the left of the graph is interpreted as a different agent with her own single-peaked preferences, we will speak of Pareto improving reallocations. There are other Pareto improving moves in the example such as $AB : 4$, $B : 10$, but there the difference between AB's and B's shares is larger, implying that such a move would lead to an allocation Lorenz dominated by our egalitarian solution $AB : 5$, $B : 9$, $A : 15$, $O : 11$.

Our Egalitarian solution shares many properties of the familiar Uniform solution in Sprumont's one supplier model, but there are important differences as well. We start with the latter.

[3] For instance, in Norway A type represent 50% of the population, while in Iceland O type represent 56% of the population (source Wikipedia).

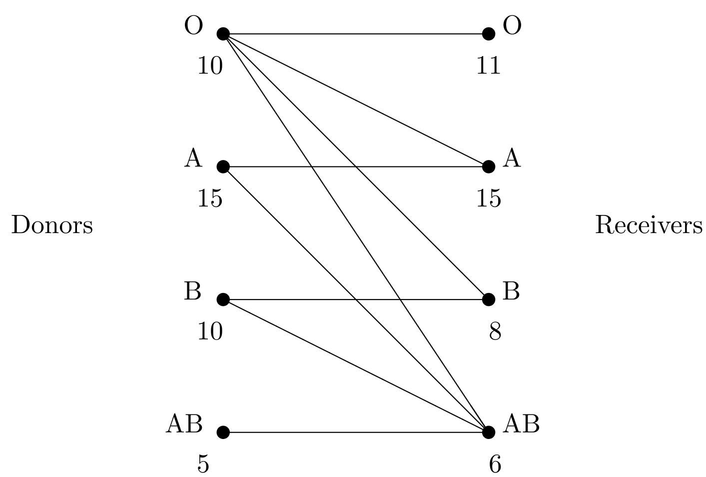

Figure 2: The donor-receiver example II

Feasibility of allocations (divisions of the resources) imposes a set of linear constraints, interpretable as the core of a concave or a convex game, and allowing us to apply the sub and supermodular optimization techniques as in (Dutta Ray 1989).

In a Pareto optimal allocation some agents always get at most their peak (the over-demanded side of the market), while some others get at least their peak (the under-demanded side of the market). In the example of Figure 2 $\{AB, B\}$ is the underdemanded side because these donors cannot meet more than 14 units of demand, but they would like to give 15 units; and $\{O\}$ is the overdemanded side because it must supply at least 11 units but would like to contribute only 10. Finally A always get exactly its peak 15 in any Pareto optimal allocation. This three-pronged partition of the agents and resources is key to our analysis. The main formal ingredient is a variant of the Gallai-Edmonds (henceforth, GE) decomposition for bipartite graphs, (see Ore (1962) for a formal treatment, or Bogomolnaia and Moulin (2004) for a matching application).

The two equity tests *No Envy* and the less demanding *Equal Treatment of Equals*, cannot be formulated as easily as in the one supplier model: given the feasibility constraints, a transfer of resources between two given agents may alter the share of a third one. We postulate that Ann envy of Bob's share is legitimate only if it is feasible to improve her a share at the expense only of Bob, i.e., while preserving the shares of every agent other than Bob. Similarly Equal Treatment of Equals is violated if we can bring Ann's and Bob's shares closer together without altering any other share.

Finally our model accommodates capacity constraints, i.e., arbitrary exogenous lower and upper bounds on each agent's allocation. This is important in all examples we discussed, where shares cannot be arbitrarily large, or small. This feature by itself is already a generalization of the standard model.

We turn to the features common to our and the one supplier model.

Our egalitarian solution is a Lorenz dominant allocation of the resources among Pareto optimal allocations; it coincides with the uniform solution if there is a single supplier. The egalitarian solution is incentive compatible in the strong sense of strategyproofness; moreover it is characterized, –as the uniform solution in (Sprumont, 1991; Ching, 1994)–, by the combination of *strategy-proofness*, *Pareto optimality* and *equal treatment of equals*.

2 Feasible allocations

We have a set M of agents with generic elements $i, j, k, \cdots$, and $m = |M|$; a set Q of resources with generic element $r, s, \cdots$, and $q = |Q|$. Resource r is of size $\omega_r, \omega_r > 0$. Agent i has capacity constraints c_i^-, c_i^+, namely his allocation x_i (share of resources) is feasible only if $c_i^- \leq x_i \leq c_i^+$. We let $[c^-, c^+] \equiv ([c_i^-, c_i^+])_{i \in M}$ be the set of capacity constraints of M.

All resources must be allocated between the agents, but each resource can only be assigned to some of the agents. The bipartite graph G, a subset of $M \times Q$, represents the compatibility constraints between resources and agents: $ir \in G$ means that it is possible to transfer resource r to agent i. We assume throughout that the graph G is connected, else we can treat each connected component of G as a separate problem.

We use the following notation: for any subsets $S \subseteq M, T \subseteq Q$ the restriction of G is $G(S,T) = G \cap \{S \times T\}$ (not necessarily connected); the set of resources compatible with agents in S is $f(S) = \{r \in Q | G(S, \{r\}) \neq \varnothing\}$, the set of agents compatible with resources in T is $g(T) = \{i \in M | G(\{i\}, T) \neq \varnothing\}$.

A transfer of resources from Q to M is described by a G-flow φ, i.e., a vector $\varphi \in \mathbb{R}_+^G$ such that $\varphi_{ir} > 0 \Rightarrow ir \in G$. We call a G-flow φ *feasible* if it allocates all the resources and we write $x(\varphi)$ for the allocation it realizes:

$$\text{for all } r \in Q: \sum_{i \in g(r)} \varphi_{ir} = \omega_j; \text{ for all } i \in M: x_i(\varphi) = \sum_{r \in f(i)} \varphi_{ir} \tag{1}$$

We write $\mathcal{F}(G;\omega)$ for the set of feasible G-flows, and $\mathcal{A}(G,\omega) = x(\mathcal{F}(G;\omega))$ for the set of allocations achieved by some feasible G-flow. Both sets are obviously non empty, but we need additional assumptions to ensure that some feasible allocations respect the capacity constraints $x \in [c^-, c^+]$. We write $x_S = \sum_{i \in S} x_i, \omega_T = \sum_{r \in T} \omega_r$ etc..

Lemma 1:*Feasible allocations* 1) *The three following statements are equivalent:*

i) $x \in \mathcal{A}(G, \omega)$;

ii) for all $S \subseteq M, x_S \leq \omega_{f(S)}$ *and* $x_M = \omega_Q$;

iii) for all $T \subseteq Q, \omega_T \leq x_{g(T)}$ *and* $\omega_Q = x_M$

2) *The set* $\mathcal{A}(G,\omega) \cap [c^-, c^+]$ *is non empty if and only if*

$$\text{for all } S \subseteq M, c_S^- \leq \omega_{f(S)}; \text{ for all } T \subseteq Q, \omega_T \leq c_{g(T)}^+ \tag{2}$$

We write $\mathcal{A}(G, \omega, c^-, c^+)$ for the set of feasible allocations respecting the capacity constraints.

Proof:

Statement 1) is a standard application of the Marriage Lemma, see [flow-textbook].

Although statement 2) is just as simple, we prove it by means of a well known[4] ascending algorithm (or a descending one) that will be used repeatedly below. Fix a continuous weakly increasing path $\lambda \in \mathbb{R}_+ \cup \{\infty\} \to \gamma(\lambda) \in \mathbb{R}_+^M$ such that $\gamma(0) = c^-, \gamma(\infty) \geq c^+$. All inequalities $\gamma_S(0) \leq \omega_{f(S)}$ hold by assumption, and $\gamma_M(\infty) \geq \omega_{f(M)}$. Let λ^1 be the largest λ such that all inequalities $\gamma_S(\lambda) \leq \omega_{f(S)}$ hold, equivalently the smallest λ such that one of them is tight. As $S \to \omega_{f(S)} - \gamma_S(\lambda^1)$ is submodular, the equality $\gamma_S(\lambda^1) = \omega_{f(S)}$ is stable by union and intersection of the sets S. Take two such subsets S, S' and compute

$$\gamma_{S \cup S'}(\lambda^1) + \gamma_{S \cap S'}(\lambda^1) = \gamma_S(\lambda^1) + \gamma_{S'}(\lambda^1) = \omega_{f(S)} + \omega_{f(S')} \geq \omega_{f(S \cup S')} + \omega_{f(S \cap S')}$$

where the latter inequality comes from $f(S \cup S') = f(S) \cup f(S')$ and $f(S \cap S') \subseteq f(S) \cap f(S')$. We call S^1 the largest such subset. By statement 1) the allocation $\gamma_{[S^1]}(\lambda^1)$ for the agents in S^1 is feasible by using all the resources in $f(S^1)$ and no more. In the restricted problem $(G(M \diagdown S^1, Q \diagdown f(S^1)), \omega)$ we have $\gamma_S(\lambda^1) \leq \omega_{f(S) \diagdown f(S^1)}$ for all $S \subseteq M \diagdown S^1$, because $\gamma_S(\lambda^1) + \gamma_{S^1}(\lambda^1) \leq \omega_{f(S) \diagdown f(S^1)} + \omega_{f(S^1)}$. In fact $\gamma_S(\lambda^1) < \omega_{f(S) \diagdown f(S^1)}$ because $\gamma_S(\lambda^1) = \omega_{f(S) \diagdown f(S^1)}$ would imply that $S^1 \cup S$ is tight at λ_1, contradicting the definition of S^1. We also have $\gamma_{M \diagdown S^1}(\infty) \geq \omega_{f(M) \diagdown f(S^1)}$, so we can repeat the argument in the restricted problem to find the smallest $\lambda, \lambda > \lambda^1$, at which one of the inequalities $\gamma_S(\lambda) \leq \omega_{f(S) \diagdown f(S^1)}$ becomes an equality. We call S^2 the largest such subset of $M \diagdown S^1$, and achieve the allocation $\gamma_{[S^2]}(\lambda^2)$ for the agents in S^2 by using all the resources in $f(S^2) \diagdown f(S^1)$ and no more. Continuing in this fashion we define a partition $S^1, S^2, \cdots$, of M, and a strictly increasing sequence $\lambda^1 < \lambda^2 < \cdots$, such that the allocation $(\gamma_{[S^1]}(\lambda^1), \gamma_{[S^2]}(\lambda^2), \cdots)$ obtains by assigning the resources in $f(S^k) \diagdown f(S^1 \cup \cdots \cup S^{k-1})$ to the agents in S^k, and meets the capacity constraints.

The descending algorithm also uses a continuous increasing path $\mu \in \mathbb{R}_+ \cup \{\infty\} \to \gamma(\mu) \in \mathbb{R}_+^M$ such that $\gamma(0) = c^-, \gamma(\infty) \geq c^+$. All inequalities $\omega_{\mathcal{T}} \leq \gamma_{g(\mathcal{T})}(\infty)$ hold by assumption, and $\omega_Q \geq \gamma_{g(Q)}(0)$. Let μ^1 be the smallest μ such that all inequalities $\omega_{\mathcal{T}} \leq \gamma_{g(\mathcal{T})}(\mu)$ hold, equivalently the largest μ such that one of them is tight; and let $\mathcal{T}^1$ be the largest set for which we have an equality (its existence guaranteed by supermodularity of $T \to \gamma_{g(T)} - \omega_T$). The allocation $\gamma_{[g(T^1)]}(\mu^1)$ to the agents of T^1 is feasible from exactly the resources in T^1. Repeating the construction in the restricted problem $(G(M \diagdown g(T^1), Q \diagdown T^1), \omega)$, etc.., we define a strictly decreasing sequence $\mu^1 > \mu^2 > \cdots$, such that the allocation $(\gamma_{[g(T^1)]}(\mu^1), \gamma_{[g(T^2)]}(\mu^2), \cdots)$ obtains by assigning the resources in

[4] It is used by {DR} to define the egalitarian solution in super/sub modular TU games, and by [BM] for the same purpose in a matching context.

T^k to the agents in $g(T^k) \diagdown g(T^1 \cup \cdots \cup T^{k-1})$, and this allocation meets the capacity constraints. This concludes the proof.■

We note for future reference that if there are no capacity constraints, $c_i^- = 0, c_i^+ = \infty$ for all i, the two algorithms define *the same* allocation if we choose the same path going up and going down. Indeed we let the reader check that if we set in the ascending algorithm $\widetilde{T}^k = f(S^k) \diagdown f(S^1 \cup \cdots \cup S^{k-1})$, and there are K subsets S^k, then $g(\widetilde{T}^K) = S^K$ and $g(\widetilde{T}^k) \diagdown g(\widetilde{T}^{k+1} \cup \cdots \cup \widetilde{T}^K) = S^k$ for all k. Thus in the descending algorithm we have $\mu^1 = \lambda^K$ and $\mu^k = \lambda^{K-k+1}$for all k.

3 Preferences and Pareto optimality

Agent i has *single-peaked preferences* over her share of resource. A single-peaked preference R_i is transitive and complete over $[c_i^-, c_i^+]$ and it has a "peak" $p[R_i] \in [c_i^-, c_i^+]$ such that for each $x_i, x_i' \in [c_i^-, c_i^+]$,

$$x_i' < x_i \leq p[R_i] \Longrightarrow x_i \; P_i \; x_i',$$
$$p[R_i] \leq x_i < x_i' \Longrightarrow x_i \; P_i \; x_i'.$$

The symmetric and asymmetric parts of R_i are denoted by I_i and P_i, respectively. Let $\mathcal{R}^i$ be the set of single-peaked preferences of agent i over $[c_i^-, c_i^+]$. A preference profile is $R = (R_i)_{i \in M} \in \mathcal{R}^M$. For each $R \in \mathcal{R}^M$, we let $p[R] = (p[R_i])_{i \in M} \in [c^-, c^+]$ be the associated profil peaks. Several of our definitions and results use only a single profile. Whenever this causes no confusion, we simply write p_i in place of $p[R_i]$. Notice that if there are no capacity constraints, i.e. $[c_i^-, c_i^+] = [0, + \propto]$ for each $i \in M$, then $\mathcal{R}^i = \mathcal{R}^j$ for each $i, j \in M$.

An allocation $x \in \mathcal{A}(G, \omega, c^-, c^+)$ is *Pareto optimal* at profile R if for any other x' in $\mathcal{A}(G, \omega, c^-, c^+)$

$$\{x_i' \; R_i \; x_i \text{ for all } i \in M\} \Rightarrow \{x_i' \; I_i \; x_i \text{ for all } i \in M\}$$

We write $\mathcal{PO}(G, \omega, c^-, c^+, R)$ for the set of Pareto optimal allocations at R.

In order to characterize Pareto optimality, we need a critical result inspired by the Gallai-Edmonds decomposition for bipartite graphs. This result depends upon G, the profile of peaks p and the profile of resources ω, but not on the capacity constraints or the other aspects of preferences.

We say that the triple (G, ω, p) is **balanced** if $p \in \mathcal{A}(G, \omega)$. We say that it exhibits **under-demand** if for all $S \subseteq M$, $p_S < \omega_{f(S)}$, and that it exhibits **over-demand** if for all $T \subseteq Q, \omega_T < p_{g(T)}$. In view of Lemma 1, in a balanced problem we can give exactly his peak allocation to every agent; in a problem with under-demand we can give each agent at least his peak, and must give to at least one strictly more; and in a problem with over-demand we can give each agent at most his peak, and must give to at least one strictly less. The decomposition below says that any allocation problem (G, ω, p) can be decomposed in three subproblems, one of each type. When we speak of the subproblem restricted to $S \times T \subseteq M \times Q$, we mean that the resources in T must be assigned to the agents in S along the restricted graph $G(S, Q)$ (which may not be possible if this graph is not connected), and we abuse notation by writing $(G(S, Q), \omega, p)$ when in fact only the S-coordinates of p and the T-coordinates of ω matter.

Lemma 2: *For any problem $(G(M, Q), \omega, p)$ where G is connected, and $p \geq 0, \omega \gg 0$, there exists unique partitions M_+, M_0, M_- of M, and Q_+, Q_0, Q_- of Q such that*
i) $G(M_-, Q_0) = G(M_-, Q_-) = G(M_0, Q_-) = \varnothing$
ii) $(G(M_+, Q_-), \omega, p)$ *exhibits under-demand;*
iii) $(G(M_0, Q_0), \omega, p)$ *is balanced;*
iv) $(G(M_-, Q_+), \omega, p)$ *exhibits over-demand.*

Note that up to two of the pairs $(M_+, Q_-), (M_0, Q_0)$, or (M_-, Q_+) may be empty. Moreover if $G(M_+, Q_-)$ (or $G(M_-, Q_+)$) is non empty, it is connected; this is not necessarily true of $G(M_0, Q_0)$.

Proof:
The Gallai-Edmonds decomposition of a bipartite graph gives precisely the statements when $p_i = \omega_r = 1$ for all i and all r. When each p_i, ω_r is a positive integer, we make p_i copies of agent i, and ω_r copies of resource r, and connect all copies of i to all copies of r iff $ir \in G$. Again the statements follow by the GE decomposition of this new bipartite graph. By a common rescaling of p and ω, we cover the case where p_i, ω_r are rational and positive, and by a straightforward limit argument that of real numbers as well, including possibly zero for some peaks.

For future reference we note that the elements of the partition can be defined as the solutions of simple maximization problems.

Define $\mathcal{D} = \arg\max_{S \subseteq M}\{p_S - \omega_{f(S)}\}$ if there is at least one S such that $p_S > \omega_{f(S)}$, $\mathcal{D} = \varnothing$ else. As $S \to p_S - \omega_{f(S)}$ is supermodular, $\mathcal{D}$ is stable by intersection and union, and M_- is its smallest element, while $M_- \cup M_0$ is its largest element. Define similarly $\mathcal{B} = \arg\max_{T \subseteq Q}\{\omega_T - p_{g(T)}\}$ if there is at least one T such that $\omega_T > p_{g(T)}$, $\mathcal{B} = \varnothing$ else. Then $\mathcal{B}$ is stable by intersection and union, Q_- is its smallest element, and $Q_- \cup Q_0$ its largest element. We omit the straightforward proof.■

There are algorithms polynomial in the number of nodes $|M| + |Q|$ to compute the GE decomposition (see Oren (1962)). In the blood donors example (see Figure 2), the GE decomposition is $M_+ = \{O\}$, $M_- = \{B, AB\}$, $M_0 = \{A\}$, $Q_- = \{O\}$, $Q_+ = \{B, AB\}$, $Q_0 = \{A\}$. For another example consider a variant of Figure 2 in which the demand from A type is 17 instead of 15. This is shown in Figure 3. There (M_+, Q_-) is the upper part of the graph while (M_-, Q_+) is the lower part. Figures 1 and 2 illustrate a general property, an immediate comsequence of Lemma 2: for any graph and any pair of agents i, j such that $f(i) \subset f(j)$, we have $\{j \in M_- \Rightarrow i \in M_-\}$ and $\{i \in M_+ \Rightarrow j \in M_+\}$.

We are now ready to describe the set $\mathcal{PO}(G, \omega, c^-, c^+, R)$ of Pareto optimal allocations, in terms of the canonical decomposition in Lemma 2. For an

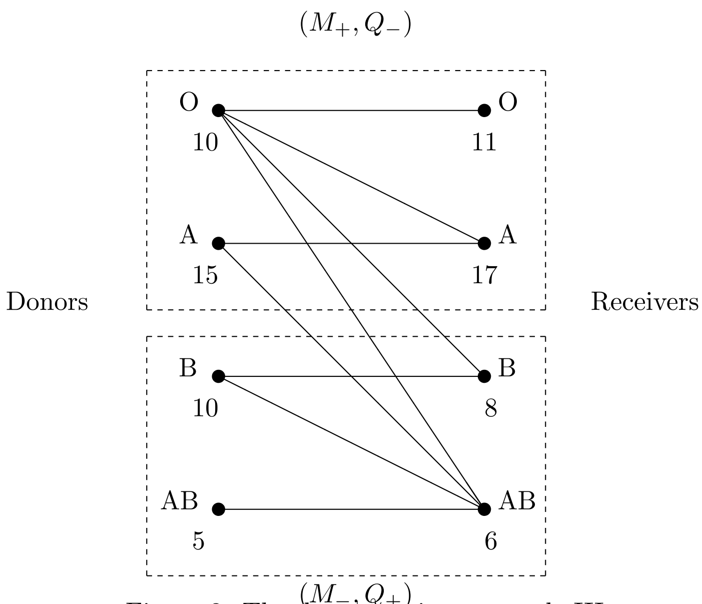

Figure 3: The donor-receiver example III

allocation $x \in \mathbb{R}^M_+$, we write its projection on $\mathbb{R}^S_+$ as $x_{[S]}$.

Proposition 1:*Assume inequalities (2) are true. For any profile $R \in \mathcal{R}^M$, the allocation x is Pareto optimal if and only if*

$$x_{[M_+]} \in \mathcal{A}(G(M_+, Q_-), \omega, p, c^+) \tag{3}$$

$$x_{[M_0]} = p_{[M_0]} \tag{4}$$

$$x_{[M_-]} \in \mathcal{A}(G(M_-, Q_+), \omega, c^-, p) \tag{5}$$

In particular the set $\mathcal{PO}(G, \omega, c^-, c^+, R)$ is non empty and the property of Pareto optimality is peak-only.

Note that by Lemma 2, the inequalities $x_{[M_+]} \geq p_{[M_+]}$ and $x_{[M_-]} \leq p_{[M_-]}$ cannot be equalities.

By Lemma 2 again, any G-flow achieving x must assign all resources in $Q_-$to agents in M_+, $Q_0$to M_0 and $Q_-$to M_+.

Proof:

Step 1 By statement 2 in Lemma 1, and the fact that $(G(M_+, Q_-), \omega, p)$ exhibits under-demand (Lemma 2), the set $\mathcal{A}(G(M_+, Q_-), \omega, p, c^+)$ is non empty; the set $\mathcal{A}(G(M_-, Q_+), \omega, c^-, p)$ is similarly non empty because $(G(M_-, Q_+), \omega, p)$ exhibits over-demand, and $\mathcal{A}(G(M_0, Q_0), \omega, p, p)$ is non empty too because $(G(M_0, Q_0), \omega, p)$ is balanced. Suppose now that an allocation x defined in (3),(4),(5) is Pareto dominated by some $y \in \mathcal{A}(G, \omega, c^-, c^+)$. Because $G(M_- \cup M_0, Q_-) = \varnothing$ we have $y_{M_+} \geq \omega_{Q_-} = x_{M_+}$; on the other hand if $y_i > x_i$ for some $i \in M_+$, this agent with peak $p_i \leq x_i$ strictly prefers x_i to y_i which our assumption precludes. We conclude $y_{[M_+]} = x_{[M_+]}$. The argument establishing $y_{[M_-]} = x_{[M_-]}$ is entirely similar.

Step 2 Conversely we fix $x \in PO(G, \omega, c^-, c^+, R)$ and a G-flow φ implementing x (system (1). To show that properties (3),(4),(5) are satisfied we proceed in 3 steps.

Step 2.1 We have

$$ir \in G(M_+, Q_0 \cup Q_+) \Rightarrow \varphi_{ir} = 0 \tag{6}$$

The proof is by contradiction. Pick $i^*r^* \in G(M_+, Q_0 \cup Q_+)$ such that $\varphi_{i^*r^*} > 0$. We construct first a *transfer path* $i^*r^*, r^*i_1, i_1r_1, r_1i_2, \cdots, r_{K-1}i_K$, entirely in $G(M_0 \cup M_-, Q_0 \cup Q_+)$ except for the first edge i^*r^*, and such that *i)* $\varphi_{i_kr_k} > 0$ for every odd edge i_kr_k; and *ii)* $x_{i_K} < p_{i_K}$. This will allow us to transfer some small amount ε of the flow $\varphi_{i^*r^*}$ to increase agent i_K's allocation without changing that of any other agent $i_1, i_2, \cdots$, along the path (or any other agent in $M_0 \cup M_-$). We simply add an ε-flow to r^*i_1, take it away from i_1r_1, add it to $r_1i_2, \cdots$, until we finally add it to $r_{K-1}i_K$. Of course ε must be smaller than the flow on any odd edge.

By Lemma 2 the set $S^1 = g(r^*) \cap (M_0 \cup M_-)$ is non empty: if $r^* \in Q_0$ then $0 < \omega_{r^*} \leq p_{g(r^*) \cap M_0}$; and if $r^* \in Q_+$ then $0 < \omega_{r^*} < p_{g(r^*) \cap M_0}$. If $x_i < p_i$ for some $i \in S^1$ then i^*r^*, r^*i is our transfer path. If $x_i \geq p_i$ for all $i \in S^1$ then $\omega_{r^*} \leq p_{S^1} \leq x_{S^1}$ but S^1 does not receive all its resources from r^* because some of ω_{r^*} goes to i^*. Therefore there exists $i_1 \in S^1$ and $r_1 \in (Q_0 \cup Q_+) \diagdown \{r^*\}$ such that $\varphi_{i_1r_1} > 0$. If there exists $i \in S^2 = g(\{r^*, r_1\}) \cap (M_0 \cup M_-)$ such that $x_i < p_i$ (note that $i \in S^2 \diagdown S^1$), the transfer path $i^*r^*, r^*i_1, i_1r_1, r_1i$ ends our construction. Else we have by Lemma 2: $\omega_{\{r^*, r_1\}} \leq p_{S^2} \leq x_{S^2}$ and as above S^2 does not receive all its resources from $\{r^*, r_1\}$. Hence there is $i_2 \in S^2$ and $r_2 \in (Q_0 \cup Q_+) \diagdown \{r^*, r_1\}$ such that $\varphi_{i_2r_2} > 0$. And so on. This process must eventually reach an agent i who receives less than his peak because $x_{M_0 \cup M_-} < \omega_{Q_0 \cup Q_+} \leq p_{M_0 \cup M_-}$ (where the first inequality is by statement *i)* in Lemma 2, and the second by statements *iii)* and *iv)*).

We now have a transfer path from the flow on i^*r^* to some agent i in $M_0 \cup M_-$ such that $x_i < p_i$. We construct similarly a transfer path from some agent j in M_+ to the flow on i^*r^*. We write it as $r^*i^*, i^*s_1, s_1j_1, j_1s_2, \cdots, s_Lj_L$, entirely in $G(M_+, Q_-)$ except for the first edge r^*i^*, and such that *i)* $\varphi_{s_lj_l} > 0$ for every odd edge s_lj_l; and *ii)* $x_{j_L} > p_{j_L}$. If $x_{i^*} > p_{i^*}$ we stop here, and if $x_{i^*} \leq p_{i^*}$ we have (Lemma 2 *ii)*) $x_{i^*} < \omega_{f(i^*) \cap Q_-}$ while all the resources $f(i^*) \cap Q_-$ must go to M_+ (Lemma 2 *i)*). Hence there is $s_1 \in f(i^*) \cap Q_-$ and $i_1 \in M_+ \diagdown i^*$ such that $\varphi_{i_1s_1} > 0$, completing the first three edges of our transfer path. And so on. The process must terminate because $x_{M_+} > \omega_{Q_-} > p_{M_+}$.

Concatenating the two paths above we can now transfer a small amount of flow from j in M_+ such that $x_j > p_j$, to i in $M_0 \cup M_-$, such that $x_i < p_i$, contradicting Pareto optimality of x.

Step 2.2 We have

$$ir \in G(M_+ \cup M_0, Q_+) \Rightarrow \varphi_{ir} = 0$$

The proof, parallel to that of step 2.1, is omitted.

Step 2.3 From step 2.1 in φ the agents in M_+ receive all the resources in Q_- and nothing else (and $x_{M_+} = \omega_{Q_-}$). Those in M_- get all the resources in Q_+ and nothing else (step 2.2), so $x_{M_-} = \omega_{Q_+}$. Thus the resources Q_0 go to agents in M_0; as $(G(M_0, Q_0), \omega, p)$ is balanced Pareto optimality requires each of these agents to obtain exactly their peak share. We show next $x_{[M_+]} \geq p_{[M_+]}$ and $x_{[M_-]} \leq p_{[M_-]}$.

Suppose $i \in M_+$ is such that $x_i < p_i$. Fix $j \in M_+$ such that $x_j > p_j$ (its existence is guaranteed by $x_{M_+} = \omega_{Q_-}$). If for some small enough $\varepsilon > 0$ the allocation x'

$$x'_i = x_i + \varepsilon; \quad x'_j = x_i - \varepsilon; \quad x'_k = x_k \text{ else} \tag{7}$$

is in $\mathcal{A}(G(M_+, Q_-), \omega)$ then it is a strict Pareto improvement of x (note that capacity constraints are automatically satisfied for x' because they are for x). Thus x' is *not* feasible in $(G(M_+, Q_-), \omega)$ for any ε however small but positive. This implies that at least one feasibility constraint bearing on x_i but not on x_j is "tight": there exists $S \subseteq M_+$ such that $i \in S, j \notin S$ and $x_S = \omega_{f(S) \cap Q_-}$. By the usual submodularity argument applied to $S \to \omega_{f(S) \cap Q_-}$, the set $\mathcal{S}(x) = \{S \subseteq M_+ | x_S = \omega_{f(S) \cap Q_-}\}$ is stable by intersection. Taking the intersection S^* of the subsets S above, for fixed i but over all j such that $x_j > p_j$, we see that $x_k \leq p_k$ for all $k \in S^*$ whereas $x_{S^*} = \omega_{f(S^*) \cap Q_-}$, a contradiction of the fact that $(G(M_+, Q_-), \omega, p)$ exhibits under-demand.

Suppose next $x_i > p_i$ for some $i \in M_-$. By the same argument as above, for any $j \in M_-$ such that $x_j < p_j$, and for any however small $\varepsilon > 0$, the perturbation of x_i, x_j as in (7) results in an unfeasible allocation x'; therefore there exists a subset S of M_- such that $x_S = \omega_{f(S) \cap Q_+}$. We now take the union $\overline{S}$ of all such subsets when i is fixed but j varies over all agents in M_- getting less than their peak; by submodularity $x_{\overline{S}} = \omega_{f(\overline{S}) \cap Q_+}$, implying $x_{M_- \diagdown \overline{S}} = \omega_{Q_+ \diagdown f(\overline{S})}$ (because $\overline{S}$ eats all the resources in $f(\overline{S}) \cap Q_+$). But by construction of $\overline{S}$, for every $k \in M_- \diagdown \overline{S}$ we have $x_k \geq p_k$, and for one of them, agent i, $x_i > p_i$. This contradiction completes the proof.■

We give below two additional examples to illustrate how the canonical decomposition works and how essential it is as a tool to compute the set of Pareto optimal allocations.

Example 1 *Load Balancing*

In Figure 4, servicepersons have to handle customers whose language may differ. Letters E, F and S stands for English, French and Spanish, respectively. The graph shows which serviceperson can communicate with which customers. The two dashedline boxes indicate the two submarkets given by the GE decomposition. In (M_+, Q_-), there is over-demand for services, while the opposite holds in (M_-, Q_+). Serviceperson S can at most serve 9 customers labeled S. In the rest of the graph, the total demand for services exceeds the total optimal workload. Serviceperson F has to serve at least 11 customers labeled F. The remaining customers E and EF will be distributed among servicepersons E and

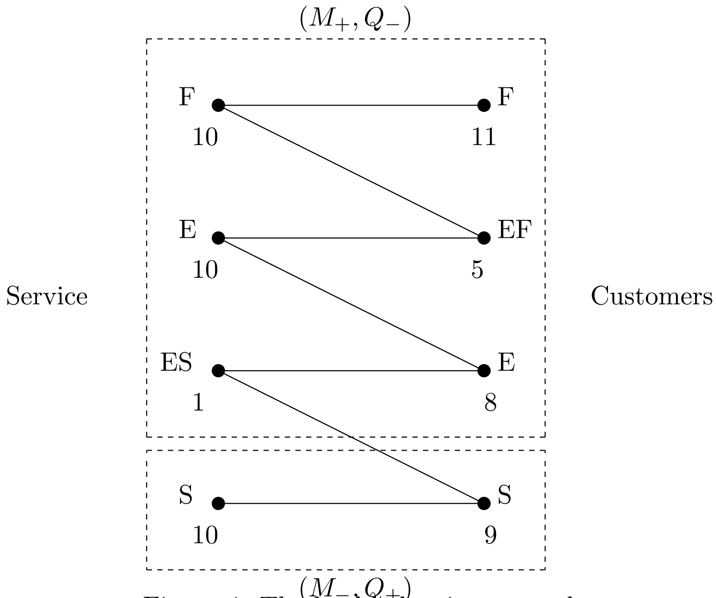

Figure 4: The load balancing example

ES. Most of the inequallities in the system (3),(4),(5) are redundant and the Pareto optimal set is given by

$$x_F + x_E + x_{ES} = 24, \; x_S = 9$$

$$11 \leq x_F, \; 10 \leq x_E, \; 1 \leq x_{ES}$$

◇

Example 2 *School assignment*

In Figure 5, S_i refer to school i and N_j refers to neighborhood j. The graph shows which schools admit students from which neighborhoods. School S_1 can admit at most 27 students, and can thus absorb all the students of neighborhoods N_1 and N_2. In the remaining of the graph, the number of students exceed the schools' peaks. Hence the students in N_3 and N_4 are distributed among S_2, S_3, and S_4. Again, the system (3),(4),(5) reduces to

$$x_{S_1} = 25, \; x_{S_2} + x_{S_3} + x_{S_4} = 35$$

$$5 \leq x_{S_2}, \; 10 \leq x_{S_3}, \; 15 \leq x_{S_4}$$

◇

4 The egalitarian solution

Notation: for any finite set N and any $z \in \mathbb{R}^N$, z^* denotes the *order statistics* of z, obtained by rearranging the coordinates of z in increasing order: $z^{*1} \leq z^{*2} \leq$

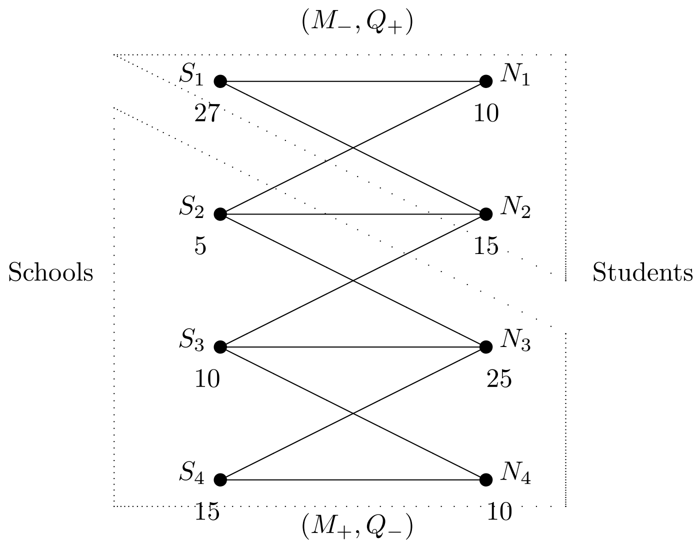

Figure 5: The school assignment example

$\cdots \leq z^{*n}$. Given two $z, w \in \mathbb{R}^N$, recall that z *Lorenz dominates* w, written z *LD* w, if for all $k, 1 \leq k \leq n$

$$\sum_{a=1}^{k} z^{*a} \geq \sum_{a=1}^{k} w^{*a}$$

We say that z is *Lorenz dominant* in the set A if z *LD* $z'$for all $z' \in A$. Lorenz dominance is a partial ordering, so not every set, even convex and compact, admits a Lorenz dominant element. On the other hand in a convex set A there can be at most one Lorenz dominant element.

The ascending/descending algorithms mentioned in our next result are defined in the proof of Lemma 1, and we use the notation $med\{a, b, c\}$ for the median of three numbers.

Proposition 2: *Assume inequalities (2) are true. For any profile* $R \in \mathcal{R}^M$*, the set of Pareto optimal allocations,* $\mathcal{PO}(G, \omega, c^-, c^+, R)$*, contains a Lorenz dominant element* $x = \mathcal{E}(R)$*, that we call the egalitarian solution. The allocation* $x_{[M_+]}$ *obtains by the ascending algorithm in* $(G(M_+, Q_-), \omega, p, c^+)$ *along the path*

$$\gamma_i^+(\lambda) = med\{\lambda, p_i, c_i^+\} \tag{8}$$

The allocation $x_{[M_-]}$ *obtains by the descending algorithm in* $(G(M_-, Q_+), \omega, c^-, p)$ *along the path*

$$\gamma_i^-(\mu) = med\{\mu, c_i^-, p_i\} \tag{9}$$

Proof:

Note first that $\gamma^+(0) = p, \gamma^+(\infty) = c^+$, so that $x_{[M_+]} \in \mathcal{A}(G(M_+, Q_-), \omega, p, c^+)$. Similarly $\gamma^-(0) = c^-, \gamma^-(\infty) = p$, so that $x_{[M_-]} \in \mathcal{A}(G(M_-, Q_+), \omega, c^-, p)$.

Proposition 1 says that the Pareto optimal set is the product of these two sets (and of the set $\{x_{[M_0]} = p_{[M_0]}\}$ in $\mathbb{R}^{M_0}$, for which we have nothing to prove). Therefore we need two independent arguments showing respectively that $x_{[M_+]}$ is Lorenz optimal in $\mathcal{A}(G(M_+, Q_-), \omega, p, c^+)$, and $x_{[M_-]}$ in $\mathcal{A}(G(M_-, Q_+), \omega, c^-, p)$. We start with the former.

Step 1. We write allocations in $\mathcal{A}(G(M_+, Q_-), \omega, p, c^+)$ simply as y, instead of $y_{[M_+]}$ and x^+ for the allocation defined by the ascending algorithm. Recall that M_+ is partitioned by $S^1, \cdots, S^k, \cdots$, such that $\gamma^+_{S^k}(\lambda^k) = \omega_{f(S^k) \diagdown f(S^1 \cup \cdots \cup S^{k-1})}$ and λ^k is strictly increasing in k. We further partition S^k as follows

$$A^k = \{i \in S^k | x_i^+ > p_i \Leftrightarrow \lambda^k > p_i\};\ B^k = \{i \in S^k | x_i^+ = p_i \Leftrightarrow \lambda^k \leq p_i\}$$

We check first that A^k is non empty for all k. By Lemma 2 *ii)*

$$p_{S^1} < \omega_{f(S^1)} = \sum_{S^1} med\{\lambda^1, p_i, c_i^+\}$$

so A^1 is non empty. Next

$$p_{S^2} \leq \gamma^+_{S^2}(\lambda^1) < \omega_{f(S^2) \diagdown f(S^1)} = \sum_{S^2} med\{\lambda^2, p_i, c_i^+\}$$

where the strict inequality is explained in the proof of Lemma 1. And so on.

Now we label the agents in M_+ in such a way that $(x^+)^{*k} = x_k^+$ for all k. We claim that in the sequence $\{1, 2, \cdots, m_+\}$

- the first $|A_1|$ terms cover A^1
- the next terms cover a possibly empty subset $\widetilde{B}^1$ of B^1
- the next $|A^2|$ terms cover A^2
- the next terms cover a possibly empty subset $\widetilde{B}^2$ of $B^1 \cup B^2$

and so on. Indeed in A^k everyone gets λ^k and the sequence λ^k increases strictly. Before A^k we cannot have any coordinate in $B^{k''}, k' \geq k$, because such an agent receives no less than λ^k.

We fix $y \in \mathcal{A}(G(M_+, Q_-), \omega, p, c^+)$ and check that it is Lorenz dominated by x^+. We use the notation $y^*(k) = \sum_{a=1}^k y^{*a}$. We have $y_S \geq y^*(|S|)$ for all S, and if $S \subset M_+$ is such that $y_S = y^*(|S|)$ we say that S is a *y-tail*.

By feasibility $y_{S^1} \leq \omega_{f(S^1)} = x^+_{S^1}$ and on the other hand $y \geq x^+$ in B^1. Therefore

$$y_S \leq x_S^+ \text{ for all } S, A^1 \subseteq S \subseteq A^1 \cup \widetilde{B}^1 \tag{10}$$

Since the above S is an x^+-tail, we have $x^{+*}(|S|) \geq y_S \geq y^*(|S|)$. Next we note that $\frac{y^*(k)}{k}$ increases weakly in k, so that for $k \leq |A_1|$ we have

$$\frac{y^*(k)}{k} \leq \frac{y^*(|A_1|)}{|A_1|} \leq \frac{y(A_1)}{|A_1|} \leq \frac{x^+(A_1)}{|A_1|} = \frac{x^{+*}(k)}{k}$$

where the equality is because x^+ is egalitarian in A^1. We have proved the desired inequality $y^*(k) \leq x^{+*}(k)$ up to $k = |A^1 \cup \widetilde{B}^1|$.

Next consider S^2: feasibility implies $y_{S^1 \cup S^2} \leq \omega_{f(S^1 \cup S^2)} = x^+_{S^1 \cup S^2}$ and $y \geq x^+$ in $B^1 \cup B^2$. Therefore

$$y_S \leq x^+_S \text{ for all } S, A^1 \cup \widetilde{B}^1 \cup A^2 \subseteq S \subseteq A^1 \cup \widetilde{B}^1 \cup A^2 \cup \widetilde{B}^2 \tag{11}$$

Again such a set S is an x^+-tail, so the inequality $y^*(k) \leq x^{+*}(k)$ follows at once for $|A^1 \cup \widetilde{B}^1 \cup A^2| \leq k \leq |A^1 \cup \widetilde{B}^1 \cup A^2 \cup \widetilde{B}^2|$. If k is such that $k = |A^1 \cup \widetilde{B}^1| + a \leq |A^1 \cup \widetilde{B}^1 \cup A^2|$, we pick $S, A^1 \cup \widetilde{B}^1 \subset S \subset A^1 \cup \widetilde{B}^1 \cup A^2$, with $|S| = k$. Because x^+ is egalitarian in A^2, we have

$$x^{+*}(k) = x^+_{A^1 \cup \widetilde{B}^1} + \frac{a}{|A^2|} x^+_{A^2} = (1 - \frac{a}{|A^2|}) x^+_{A^1 \cup \widetilde{B}^1} + \frac{a}{|A^2|} x^+_{A^1 \cup \widetilde{B}^1 \cup A^2}$$

We claim

$$y^*(k) \leq y_{A^1 \cup \widetilde{B}^1} + \frac{a}{|A^2|} y_{A^2} = (1 - \frac{a}{|A^2|}) y_{A^1 \cup \widetilde{B}^1} + \frac{a}{|A^2|} y_{A^1 \cup \widetilde{B}^1 \cup A^2} \tag{12}$$

which will imply $y^*(k) \leq x^{+*}(k)$ because $y_S \leq x^+_S$ is true both for $A^1 \cup \widetilde{B}^1$ and $A^1 \cup \widetilde{B}^1 \cup A^2$.

The claim follows from the following fact: if X, Y, Z are three disjoint subsets, we have

$$y^*(|X| + |Y|) \leq y_X + \frac{|Y|}{|Y| + |Z|} y_{Y \cup Z}$$

Indeed $\frac{|Y|}{|Y|+|Z|} y_{Y \cup Z}$ is no less than the sum of the $|Y|$ lowest terms in $y_{[Y \cup Z]}$, and y_X is no less than the sum of the $|X|$ lowest terms in $y_{[X]}$. Applying this inequality to $X = A^1 \cup \widetilde{B}^1, Y = S \diagdown (A^1 \cup \widetilde{B}^1)$ and $Z = (A^1 \cup \widetilde{B}^1 \cup A^2) \diagdown S$ gives (12).

Step 2. In $\mathcal{A}(G(M_-, Q_+), \omega, c^-, p)$ the descending algorithm defining x^- yields a partition $S^1, \cdots, S^k, \cdots,$ of M_-, and a strictly decreasing sequence μ^k such that $\omega_{T^k} = \gamma^-_{g(T^k) \diagdown g(T^1 \cup \cdots \cup T^{k-1})}(\mu^k)$. We set $S^k = g(T^k) \diagdown g(T^1 \cup \cdots \cup T^{k-1})$ so that in x^- the resources in T^k are assigned to agents in S^k. Then we partition S^k as follows

$$A^k = \{i \in S^k | x^-_i < p_i \Leftrightarrow \mu^k < p_i\}; \ B^k = \{i \in S^k | x^-_i = p_i \Leftrightarrow \mu^k \geq p_i\}$$

The set A^1 is non empty because $\sum_{S^1} med\{\mu^1, c^-_i, p_i\} = \omega_{T^1} < p_{S^1}$, A^2 is non empty because

$$p_{S^2} \geq \gamma^-_{S^2}(\mu^1) > \omega_{T^2} = \sum_{S^2} med\{\mu^2, c^-_i, p_i\}$$

and so on. Labeling the agents in M_- so that $(x^-)^{*k} = x^-_k$ for all k, we see that in the sequence $\{m_-, m_- - 1, \cdots, 1\}$

- the first $|A_1|$ terms cover A^1
- the next terms cover a possibly empty subset $\widetilde{B}^1$ of B^1
- the next $|A^2|$ terms cover A^2
- the next terms cover a possibly empty subset $\widetilde{B}^2$ of $B^1 \cup B^2$

and so on. This is because a coordinate in A^k receives μ^k and one in B^k no more than μ^k.

For an arbitrary $y \in \mathcal{A}(G(M_-, Q_+), \omega, c^-, p)$ we use the notation $y^*(k) = \sum_{a=m_-}^{m_- - k+1} y^{*a}$.so that $y_S \leq y^*(|S|)$ for all S. The end of the proof that y is Lorenz dominated by x^- is entirely similar to the one in step 1, upon reversing the direction of inequalities. That is, the feasibility constraints $\omega_T \leq y_{g(T)}$ imply now $y_{S^1} \geq \omega_{T^1} = x^-_{S^1}$; on the other hand $y \leq x^-$ in B^1, and so (10) follows (up to a change of sign). Similarly the inequality $y_{S^1 \cup S^2} \geq \omega_{T^1 \cup T^2} = x^-_{S^1 \cup S^2}$ holds by feasibility of y etc..■

Example 3 *Want all or nothing Suppose there are no capacity constraints, $c_i^- = 0, c_i^+ = \infty$. Then the egalitarian solution when all peaks are zero (resources are all "bad"), is the same as when all are infinite (resources are all "good"). Indeed in the former case the GE decomposition (Lemma 2) reduces to (M_+, Q_-) and the ascending algorithm follows the path $\gamma_i^+(\lambda) = \lambda$. In the latter case, the decomposition is just (M_-, Q_+) and the descending algorithm follows the same path $\gamma_i^-(\lambda) = \lambda$. The claim follows from the remark following Lemma 1. The egalitarian solution is precisely the Dutta-Ray egalitarian solution for the concave TU game $(M, v) : v(S) = \omega_{f(S)}$.* ◇

Example 4 *Load Balancing revisited I*

Since only $S \in M_-$, we have $x_S = 9$. For (M_+, Q_-) the ascending algorithm stops at $\mu_1 = 3$ where $\max\{p_E, 3\} + \max\{p_{ES}, 3\} = 13 = \omega_{f(E,ES)}$, then at $\mu_2 = 11$ where $\max\{p_F, 11\} = \omega_{f(F) \setminus f(E,ES)}$. So $x_E = 3, x_{ES} = 10, x_F = 11$. ◇

Example 5 *School assignment revisited I*

Since only $S_1 \in M_-$, we have $x_{S_1} = 25$. For (M_+, Q_-), we get $\mu_1 = 10$ as $\sum \max\{p_i, 10\} = 35$ for $i = S_2, S_3, S_4$, so that $x_{S_2} = x_{S_3} = 10, x_{S_4} = 15$. Notice the lack of "progressivity" of the egalitarian solution –a consequence of strategy-proofness– since S_2 is imposed to admit a number of students that exceeds by far its peak while S_3 is at its peak. ◇

5 Properties of the egalitarian rule

We define a handful of properties that the egalitarian *rule* satisfies.

Definition: *Given the resources, agents, feasibility and capacity constraints (G, ω, c^-, c^+) a rule selects for every preference profile $R \in \mathcal{R}^M$ a feasible allocation $\psi(R) \in \mathcal{A}(G, \omega, c^-, c^+)$.*

We start with the familiar equity test of No Envy, that must be adapted to our model because of the feasibility constraints. If agent 1 envies the allocation x_2 of agent 2, it might not be possible anyway to give him x_2 because the resources connected to agent 1 do not allow it. Moreover unlike in a classic fair division problem where shares can be exchanged between 1 and 2 without affecting other agents' shares, here exchanging the allocations of 1 and 2 typically requires to construct a new flow, which may force us to alter some of the other agents' allocations.

In a case like the one above, we posit that agent 1's claim has no legitimate claim against the allocation just described. We view an envy argument by agent 1 against agent 2 as legitimate only if it is feasible to improve upon agent 1's allocation without altering the allocation of anyone other than agent 2.

No Envy: *A rule ψ satisfies No Envy if for each $R \in \mathcal{R}^M$ and $\{i, j\} \subset M$ such that $\psi_j(R)\, P_i\, \psi_i(R)$, there exists no $x \in A(G, \omega, c^-, c^+)$ such that*

$$\psi_k(R) = x_k \text{ for each } k \neq i, j \text{ and } x_i\, P_i\, \psi_i(R)$$

In the standard single resource model without capacity constraint, this definition is the standard one (Sprumont, 1991).

Example 6 *Load balancing revisited II*

Going back to Figure 4, consider the efficient allocation $(x_F, x_E, x_{ES}, x_S) = (11, 12, 1, 9)$. *Suppose that preferences of E, R_E are such that $1\ P_E\ 12$. So E envies ES. Notice that this claim is legitimate because we can improve upon E's bundle without altering the bundle of anyone else's than ES. Indeed, the allocation* $(11, 11, 2, 9)$ *is such an allocation. If R_E is such that $2\, P_E\, 11$, then again the envy claim is legitimate because of the allocation* $(11, 10.5, 2.5, 9)$. *Repeating the argument, we see that E stops having a legitimate envy claim on ES only at allocation* $(11, 10, 3, 9)$. *Notice also that in each of these allocations, F envies E. But this claim is not legitimate since no transfer between E and F can be performed to reduce the envy level of F. One checks that there exists a unique allocation which passes the no-envy test, and it is the egalitarian allocation* $(11, 10, 3, 9)$. $\diamond$

The basic horizontal equity property known as equal treatment of equals must also be adapted to take feasibility constraints into account. If two agents have incompatible capacity constraints (e.g., $c_1^+ < c_2^-$), they cannot have the same preferences, and so they cannot be "equals" in the usual sense. We shall say that the two preferences $R_i \in \mathcal{R}^i, R_j \in \mathcal{R}^j$ are "constrained-equal" if

$$\text{either } \{p[R_i] = p[R_j]\} \text{ or } \{\{p[R_i] < p[R_j]\} \text{ and } \{p[R_i] = c_i^+ \text{ and/or } p[R_j] = c_j^-\}\} \quad (13)$$

or a symmetric configuration by exchanging the role of i and j. If $p[R_i] = c_i^+ < p[R_j]$, agent j's preferences "project" on $[c_i^-, c_i^+]$ to precisely agent i's preferences, whether or not some shares in $[c_i^-, c_i^+]$ are compatible with j's capacity constraints.

We now require that the allocations of two agents with "constrained-equal" preferences should be as close as possible, given that we cannot alter anyone else's allocation.

Equal Treatment of Equals: *A rule ψ satisfies ETE if for each $R \in \mathcal{R}^M$ and $\{i,j\} \subset M$ such that R_i and R_j are constrained-equal and $\psi_j(R) \neq \psi_i(R)$, there exists no $x \in A(G, \omega, c^-, c^+)$ such that*

$$\psi_k(R) = x_k \text{ for each } k \neq i,j \text{ and } |x_i - x_j| < |\psi_j(R) - \psi_i(R)|$$

Like in the one resource model, here ETE is a consequence of No Envy.

Proposition 3: *The egalitarian rule satisfies No Envy.*
This follows at once from the fact that the egalitarian allocation is Lorenz dominant, and the set of feasible allocations is convex.

We turn to the basic incentive property of strategyproofness. As in the one resource model, we can decompose it into a monotonicity and an invariance condition. Here, we use the notation $p[R_i]$ for the peak of the single-peaked preference R_i.

Monotonicity: *A rule ψ is monotonic if for all $R \in \mathcal{R}^M$, $i \in M$, and $R'_i \in \mathcal{R}^i$*

$$p[R'_i] \leq p[R_i] \Rightarrow \psi_i(R'_i, R_{-i}) \leq \psi_i(R)$$

Invariance: *A rule ψ is invariant if for all $R \in \mathcal{R}^M$, $i \in M$, and $R'_i \in \mathcal{R}^i$*

$$\{p[R_i] < \psi_i(R) \text{ and } p[R'_i] \leq \psi_i(R)\} \Rightarrow \psi_i(R'_i, R_{-i}) = \psi_i(R) \qquad (14)$$

$$\{p[R_i] > \psi_i(R) \text{ and } p[R'_i] \geq \psi_i(R)\} \Rightarrow \psi_i(R'_i, R_{-i}) = \psi_i(R)$$

Strategyproofness: *A rule ψ is strategyproof if for all $R \in \mathcal{R}^M$, $i \in M$, and $R'_i \in \mathcal{R}^i$*

$$\psi_i(R) R_i \psi_i(R'_i, R_{-i})$$

The next Lemma connects these three properties and Pareto optimality.

Lemma 3: *Monotonicity and invariance*
i) If a rule is monotonic and invariant, it is strategy-proof;
ii) An efficient and strategyproof rule is monotonic and invariant.

Proof:
We omit the easy argument proving statement *i*), just as in the one resource model.
Statement ii) Fix an efficient (Pareto optimal) and strategyproof rule ψ. We show first that the mapping $R_i \to \psi_i(R_i, R_{-i})$ is peak-only. Fix R_{-i} and consider two preferences R_i, R'_i such that $p[R_i] = p[R'_i]$. The GE decomposition (Lemma 2) is the same in R and (R'_i, R_{-i}), so by efficiency agent i's allocations x_i and x'_i are on the same side of $p[R_i]$. Now strategyproofness implies peak-only.

Next we prove monotonicity. We fix i, R *and* R'_i such that $p'_i = p[R'_i] \leq p[R_i] = p_i$, and let p, p' be the profile of peaks at R and (R'_i, R_{-i}) respectively. We also set $x_i = \psi_i(R), x'_i = \psi_i(R'_i, R_{-i})$. Distinguish two cases.

Case 1: $i \in M_-(p)$. Assume first $p'_i > x_i$.Then the decomposition is unchanged, in particular $M_-(p) = M_-(p')$, so by efficiency $x'_i \leq p'_i$. Assume $x_i < x'_i$; then we have $x_i < x'_i \leq p'_i \leq p_i$, and we get a contradiction of SP for agent i at profile R. Assume next $p'_i \leq x_i$. Then $x_i < x'_i$ would give $p'_i \leq x_i < x'_i$, hence a violation of SP for agent i with preference R'_i.

Case 2: $i \in (M_0 \cup M_+)(p)$. Then $p_i \leq x_i$, so $x_i < x'_i$ would give $p'_i \leq p_i \leq x_i < x'_i$, hence a violation of SP for agent i at R'_i.

We show finally that ψ is invariant. Under the premises of property (14), if $\psi_i(R'_i, R_{-i}) > \psi_i(R)$ we have $p[R'_i] \leq \psi_i(R) < \psi_i(R'_i, R_{-i})$, hence a violation of SP for agent i at R'_i. If $\psi_i(R'_i, R_{-i}) < \psi_i(R)$ we can find a preference R^*_i such that $p[R^*_i] = p_i[R_i]$ and $\psi_i(R'_i, R_{-i}) P^*_i \psi_i(R)$. By peak-onliness, $\psi_i(R^*_i, R_{-i}) = \psi_i(R)$, so agent i with preferences R^*_i benefits by reporting R'_i. The proof of the second property is identical.■

Proposition 4: *The egalitarian rule is monotonic and invariant, hence strategyproof as well.*

Proof:

We fix an agent i and a benchmark profile of peaks p, with corresponding egalitarian allocation x. We consider changes of peak only by agent i to p'_i, and we write $p'_j = p_j$ for all $j \neq i$, so that $p' = (p'_i, p_{-i})$.

Step 1a Suppose $i \in M_+(p)$. Then if $p'_i < p_i$ the GE decomposition (Lemma 2) does not change, so $i \in M_+(p')$. Consider the critical report p^*_i, $p^*_i > p_i$, if any, at which the GE decomposition and the status of agent i change.. By Lemma 2 *ii*), there is no change at p'_i as long as $p'_S < \omega_{f(S) \cap Q_-(p)}$ for all $S \subseteq M_+(p)$. Thus p^*_i is the smallest number such that

$$p_{S \diagdown i} + p^*_i = \omega_{f(S) \cap Q_-(p)} \tag{15}$$

for some subset S of $M_+(p)$ containing i. If $p^*_i > c^+_i$, then in fact the decomposition never changes when p'_i varies in the relevant interval, and we set $p^*_i = \infty$ to fix ideas. Assume from now on $p^*_i \leq c^+_i$, and let S^* be the largest S satisfying (15) (well defined by the usual submodularity argument). Recall from the proof of Lemma 2 that $(M_- \cup M_0)(p)$ is the largest solution of $\arg\max_{S \subseteq M}\{p_S - \omega_{f(S)}\}$. At p^* we have $\max_S\{p^*_S - \omega_{f(S)}\} = \max_S\{p_S - \omega_{f(S)}\}$ and the largest solution of $\arg\max_{S \subseteq M}\{p^*_S - \omega_{f(S)}\}$ is now $(M_- \cup M_0)(p) \cup S^* = (M_- \cup M_0)(p^*)$; moreover $M_-(p)$ is still a solution of $\arg\max_{S \subseteq M}\{p^*_S - \omega_{f(S)}\}$, therefore it is the smallest. So $i \in M_0(p^*)$.

Moreover $x_{[M_+]} \in \mathcal{A}(G(M_+, Q_-), \omega, p, c^+)$ (Pareto optimality) and $S^* \subseteq M_+(p)$ together imply

$$x_{S^*} \leq \omega_{f(S^*) \cap Q_-(p)}$$

We have $p_{[M_+]} \leq x_{[M_+]}$, so if $p^*_i < x_i$, we would have $p^*_{S^*} < x_{S^*}$, and a contradiction of (15) for S^*. Therefore

$$p^*_i \geq x_i \geq p_i \tag{16}$$

Step 1b Suppose $i \in M_-(p)$. By entirely symmetric arguments we can show that one of two cases arises. If $M_-(c^-_i, p_{-i}) = M_-(p)$, the decomposition never changes when p'_i varies in the relevant interval, and we set $p^*_i = -\infty$. Otherwise there is a critical peak p_i^* below p_i at which the decomposition changes for the first time. The details of the decomposition at p^* are similar and they only matter to prove $i \in M_0(p^*)$ and

$$p_i^* \leq x_i \leq p_i \tag{17}$$

Step 2a Consider a change of peak from p_i to p'_i such that $i \in M_+(p) = M_+(p')$.
Suppose first $p'_i < p_i$. We show $x'_i \leq x_i$ by distinguishing two cases. Write in both cases S^k, λ^k for the partition and corresponding parameters of the ascending algorithm at p, and let $i \in S^\ell, x_i = \lambda^k \vee p_i$.

First case: $p_i < \lambda^\ell = x_i$. Then the partition and corresponding parameters are unchanged at p' so that $x'_i = x_i$.

Second case: $p_i = x_i \geq \lambda^\ell$. Then S^k, λ^k are unchanged for $1 \leq k \leq \ell - 1$, but S^ℓ, λ^ℓ may change. However for $\lambda = p_i$

$$\sum_{S^l} \lambda \vee p'_j \geq \sum_{S^l} \lambda^\ell \vee p'_j = \omega_{T^l}$$

(where we write $T^l = f(S^l) \diagdown f(S^1 \cup \cdots \cup S^{l-1})$), therefore if S^ℓ changes, the new set $\widetilde{S}\ell$ contains i and $\widetilde{\lambda}^\ell \leq p_i$, hence $x'_i \leq p_i = x_i$.
Suppose next, until the end of step 2a, $p_i < p'_i$. If $p'_i \geq x_i$ notice that $i \in M_+(p')$ implies $x'_i \geq p'_i$ so we are done. So we are left with the case $p_i < p'_i < x_i = \lambda^\ell$, that requires more work. We prove by induction on ℓ that the first ℓ terms S^k, λ^k,$1 \leq k \leq \ell$, of the partition and corresponding parameters are unchanged at p'. We write $\widetilde{S}^k$, $\widetilde{\lambda}^k$ for the latter.

Suppose $\ell = 1$, then $\sum_{j \in S} \mu^1 \vee p_j = \sum_{j \in S} \mu^1 \vee p'_j$ for all $S \subseteq M_+(p)$, so the claim holds.

Next suppose $\ell \geq 2$. Assume $S^1 \neq \widetilde{S}^1$ and derive a contradiction. This implies there exists a coalition $S \subseteq M_+(p)$ such that $S \subsetneq S^1$ and

$$\sum_{j \in S} \lambda^1 \vee p'_j \geq \omega_{f(S)} \tag{18}$$

Indeed suppose (18) fails for all $S \subsetneq S^1$: as p and p' coincide inside S^1, we would get $S^1 = \widetilde{S}^1$. Fix a coalition S as in (18), that must contain i, hence $S \cap S^\ell$ is non empty. By definition of the ascending algorithm, the sets $T^1 = f(S \cap S^1), \cdots, T^k = f(S \cap S^k) \diagdown (T^1 \cup \cdots \cup T^{k-1}), \cdots$,are pairwise disjoint and $\sum_{S \cap S^k}^k \lambda \vee p_j \leq \omega_{T^k}$ for all k, therefore

$$\sum_{1 \leq k \leq K} \left[\sum_{S \cap S^k} \lambda^k \vee p_j \right] \leq \omega_{f(S)}$$

In view of (18), we get

$$\sum_{1 \leq k \leq K} \left[\sum_{S \cap S^k} \lambda^k \vee p_j \right] \leq \sum_{j \in S} \lambda^1 \vee p'_j$$

For all $k \neq \ell$, we have $\lambda^k \geq \lambda^1$ and $p = p'$ in $S \cap S^k$, implying $\sum_{S\cap S^k} \lambda^k \vee p_j \geq \sum_{S\cap S^k} \lambda^1 \vee p'_j$. As λ^ℓ is larger than λ^1, p'_i, and p_i, and $S \cap S^\ell$ is non empty, we get $\sum_{S\cap S^\ell} \mu^\ell \vee p_j > \sum_{S\cap S^\ell} \mu^1 \vee p'_j$. The desired contradiction follows and we conclude $S^1 = \widetilde{S}^1$.

To show next $S^2 = \widetilde{S}^2$, we replicate the above argument as follows. If $\ell = 2$, then $\sum_{j\in S} \lambda^2 \vee p_j = \sum_{j\in S} \lambda^2 \vee p'_j$ for all $S \subseteq M_+(p)\diagdown S^1$, because $p_i, p'_i < \lambda^2$, and the claim holds. If $\ell \geq 3$ and $S^2 \neq \widetilde{S}^2$, we can pick a coalition $S \subseteq M_+(p)\diagdown S^1$ such that $S \subsetneq S^2$ and

$$\sum_{j\in S} \lambda^2 \vee p'_j \geq \omega_{f(S)\diagdown S^1}$$

and proceed as above by decomposing S along $S^k, 2 \leq k \leq K$. The induction step is now clear.

Step 2b For a change of peak from p_i to p'_i such that $i \in M_-(p) = M_-(p')$, the parallel argument is omitted for brevity.

Step 3 In step 2 we proved monotonicity for shifts in p_i inside $M_+(p)$ or inside $M_-(p)$. Consider now a move from p_i to p'_i when $i \in M_0(p)$. If $p'_i > p_i$, we have $i \in M_-(p')$. Then in the downward shift starting at p'_i, p_i is the critical value at which the status of i changes, so by (17) $x'_i \geq p_i = x_i$ as desired. Symmetrically $p'_i < p_i$ gives $i \in M_+(p')$ and p_i is the critical value starting from p'_i described in step 1a, so (16) gives $x'_i \leq p_i = x_i$.

It remains to look at a shift from p_i to p'_i such that $i \in M_+(p)$ and $i \in M_-(p')$. This requires $p'_i > p_i$ and the critical value p^*_i for p_i described in step 1a is the same as the critical value for p'_i in step 1b. Therefore (16) and (17) imply

$$p_i \leq x_i \leq p^*_i \leq x'_i \leq p'_i$$

Step 4 The invariance property is clear from (16) and (17) and the arguments of step 2a, 2b.■

6 Characterization result

Theorem 1: *The egalitarian rule is characterized by Pareto optimality, strategyproofness and equal treatment of equals.*

Proof

The proof is inspired by that in Ching (1994). We fix G, ω, c^-, c^+, a rule ψ meeting the three properties. To simplify notations, for any $R \in \mathcal{R}^M$ with profile of peaks $p \in [c^-, c^+]$, we write the M_+-component of the Pareto optimal set as the core$^+$ of a concave (submodular) cooperative game (M_+, v^+) and the M_--component as the core$^-$ of a convex (supermodular) game. Here $Core^+$ refers to the system of inequalities $y_S \leq v^+(S)$ in M_+, and $Core^-$ to the system $y_S \geq v^-(S)$ in M_-.Define

$$v^+(S) = \omega_{f(S)\cap Q_-} \text{ for all } S \subseteq M_+$$

$$v^-(S) = \max\{\omega_T | T \subseteq Q_+, g(T) \cap M_- \subseteq S\} \text{ for all } S \subseteq M_-$$

From Lemma 1 statement 1) and Proposition 1 we deduce easily $\mathcal{A}(G(M_+, Q_-), \omega, p, c^+) = Core^+(M_+, v^+) \cap [p, c^+]$, and $\mathcal{A}(G(M_-, Q_+), \omega, c^-, p) = Core^-(M_-, v^-) \cap [c^-, p]$. Checking that these games are respectively concave and convex is also straightforward.

By Propositions 1 and 2 it is enough to prove two separate statements for any profile R,: $\psi(R)_{[M_+]}$ is Lorenz dominant (egalitarian) in $Core^+(M_+, v^+) \cap [p, c^+]$, and $\psi(R)_{[M_-]}$ is similarly Lorenz dominant in $Core^-(M_-, v^-) \cap [c^-, p]$. There is nothing to prove for the agents in M_0.

Step 1 Suppose R is a profile of pairwise constrained-equal preferences (13). Let $y = \psi(R)$ and write its M_+ component simply as y. We invoke only ETE (and Pareto optimality) to show that $y = x$, the Lorenz dominant allocation in $Core^+(M_+, v^+) \cap [p, c^+]$.

Proposition 1 and ETE imply that for any pair $i, j \in M_+$ such that $y_i \neq y_j$ and any $y' \in \mathbb{R}^{M_+}$

$$\{y'_{M_+} = y_{M_+},\ |y'_i - y'_j| < |y_i - y_j|,\ y'_k = y_k \text{ for all } k \neq i, j\} \Rightarrow y' \notin Core^+(M_+, v^+) \tag{19}$$

Indeed if we could find y' in $Core^+(M_+, v^+)$ satisfying the premises of (19), the constraints $y' \in [p, c^+]$ would be satisfied so it would be feasible, without altering the allocation of any other agent, to perform a Pigou-Dalton transfer between agents i, j reducing the difference in their shares. This is ruled out by ETE.

Claim 1 Fix an agent $i_1 \in M_+$, such that $y_{i_1} = y^{*1}$, then

$$y_{i_1} = x_{i_1} = y^{*1} = x^{*1} \tag{20}$$

Because x Lorenz dominates y, we have $y^{*1} \geq x^{*1}$. If $y_i = y^{*1}$ for all $i \in M_+$ then $y = x$ at once, so we can assume there is at least one agent such that $y_i < y^{*1}$. For any such agent there must exist a subset $S(i) \subset M_+$ such that

$$i_1 \notin S(i),\ i \in S(i),\ \text{and } y_{S(i)} = v^+(S(i))$$

Otherwise $y_S < v^+(S)$ for all S in M_+ containing i but not i_1. Choosing $\varepsilon > 0$ smaller than the smallest such difference $v^+(S) - y_S$, we see that an ε-transfer from agent i_1 to agent i (a Pigou-Dalton transfer) preserves the core property (inequalities $y_S \leq v^+(S)$ for S containing i are automatically satisfied), in contradiction of (19).

We set $S^* = \cup_{i:y_i<y^{*1}} S(i)$. By submodularity of v^+ we have $y_{S^*} = v^+(S^*)$ and by construction $y_j = y^{*1}$ for all $j \in N \diagdown S^*$; furthermore $N \diagdown S^*$ contains i_1. We have

$$x_{S^*} \leq v^+(S^*) = y_{S^*} \Rightarrow x_{N \diagdown S^*} \geq y_{N \diagdown S^*} = |N \diagdown S^*| \cdot y^{*1}$$

Combining this with $y^{*1} \geq x^{*1}$ proves (20).

Claim 2 Fix an agent $i_2 \in M_+$, such that $i_2 \neq i_1$ and $y_{i_1} = y^{*1}$, then

$$y_{i_2} = x_{i_2} = y^{*2} = x^{*2} \tag{21}$$

As x Lorenz dominates y, we have $y^{*1} + y^{*2} \geq x^{*1} + x^{*2} \Rightarrow y^{*2} \geq x^{*2}$. If $y_i = y^{*2}$ for all $i \in M_+ \diagdown i_1$ then $y \geq x$ so $y = x$ by $y_{M_+} = x_{M_+}$, and we are done. Now we assume there is at least one agent such that $y_i < y^{*2}$. For any such agent there is a subset $S(i) \subset M_+$ such that

$$i_2 \notin S(i),\ i \in S(i), \text{ and } y_{S(i)} = v^+(S(i))$$

Otherwise, as above we can construct a Pigou-Dalton transfer from agent i_2 to agent i. Set $S^* = \cup_{i:y_i<y^{*2}} S(i)$, then submodularity of v^+ gives $y_{S^*} = v^+(S^*)$, moreover $y_j \geq y^{*2}$ for all $j \in N \diagdown S^*$ hence

$$x_{S^*} \leq v^+(S^*) = y_{S^*} \Rightarrow x_{N \diagdown S^*} \geq y_{N \diagdown S^*} \Rightarrow x_{N \diagdown (S^* \cup \{i_1\})} \geq y_{N \diagdown (S^* \cup \{i_1\})}$$

For each j in the non empty set $N \diagdown (S^* \cup \{i_1\})$ (that contains i_2), we have $x_j \leq x^{*2}$ and $y_j \geq y^{*2}$. In view of $y^{*2} \geq x^{*2}$, all these inequalities are equalities and (21) is proven. The inductive argument establishing $y = x$ is now clear.

We omit for brevity the entirely similar argument establishing step 1 for the M_--component of $\psi(R)$.

Step 2a Notation: if $R, \widetilde{R} \in \mathcal{R}^M$ and $S \subset M$, we write $(R[S], \widetilde{R}[M \diagdown S])$ the profile equal to R for agents in S and to $\widetilde{R}$ for agents in $M \diagdown S$. We also write ψ^e for the egalitarian rule.

Fix a profile $\widetilde{R}$ of pairwise constrained-equal preferences, an integer $n, 0 \leq n \leq m-1$ and consider the following subset of preference profiles

$$R \in \mathcal{D}(\widetilde{R}, n) \overset{def}{\Leftrightarrow}$$

$$\text{for some } S \subset M : |S| \leq n \text{ and } R = (R[S], \widetilde{R}[M \diagdown S]); \text{and } p[\widetilde{R}_i] \leq p[R_i] \text{ if } i \in M_+(R)\}$$

We prove by induction on n the following property $\mathcal{H}^+(n)$: for all $R \in \mathcal{R}^M$ and all $S \subset M$

$$R \in \mathcal{D}(\widetilde{R}, n) \Rightarrow \psi_i(R) = \psi_i^e(R) \text{ for all } i \in M_+(R)$$

Step 1 establishes $\mathcal{H}^+(0)$. Assume now $\mathcal{H}^+(n-1)$ is true, and fix $R = (R[S], \widetilde{R}[M \diagdown S]) \in \mathcal{D}(\widetilde{R}, n)$ with $|S| = n$. We pick an agent $i \in S \cap M_+(R)$, so by Pareto optimality

$p[R_i] \leq \psi_i(R), \psi_i^e(R)$. To prove $\psi_i(R) = \psi_i^e(R)$ we consider the profile $R' = (R[S \diagdown i], \widetilde{R}[(M \diagdown S) \cup \{i\}]) \in \mathcal{D}(\widetilde{R}, n-1)$ where by the inductive assumption we have $\psi_i(R') = \psi_i^e(R') = z_i$. We compare $\psi_i(R), \psi_i^e(R)$ and z_i by distinguishing two cases.

If $p[R_i] \leq \psi_i(R) < \psi_i^e(R)$ then $z_i \leq \psi_i(R)$ by monotonicity of ψ (Lemma 3) in the shift from R_i to $\widetilde{R}_i$, and $z_i = \psi_i^e(R)$ by invariance of ψ^e (Lemma 3), a contradiction. If $p[R_i] \leq \psi_i^e(R) < \psi_i(R)$ the same contradiction obtains by exchanging the role of ψ and ψ^e.

We just proved $\psi_i(R) = \psi_i^e(R)$ for $i \in S \cap M_+(R)$, and it remains to check it for $M_+(R) \diagdown S$ as well. Write simply $M_+(R) = M_+$, $\psi_i(R) = y$, $\psi^e(R) = x$, and define the set

$$\mathcal{C}(R) = \{z \in \mathbb{R}_+^{M_+ \diagdown S} | (z, x_{[S \cap M_+]}) \in Core^+(M_+, v^+) \cap [p, c^+]\}$$

Clearly $x_{[M_+ \diagdown S]}$ is still Lorenz dominant in $\mathcal{C}(R)$, hence we can mimic the proof of Step 1 to show that ETE and Pareto optimality imply the desired equality of y and x in $M_+ \diagdown S$. The key is that the profile $\widetilde{R}[M_+ \diagdown S]$ consists of pairwise constrained-equal preferences, therefore we can apply ETE to any pair of agents in $M_+ \diagdown S$. Then Proposition 1 and ETE imply that for any pair $i, j \in M_+ \diagdown S$ such that $y_i \neq y_j$ and any $y' \in \mathbb{R}^{M_+}$

$$\{y'_{M_+ \diagdown S} = y_{M_+ \diagdown S},\ |y'_i - y'_j| < |y_i - y_j|,\ y'_k = y_k \text{ for all } k \neq i, j\} \Rightarrow y' \notin \mathcal{C}(R)$$

To copy the proof of step 1, observe that $\mathcal{C}(R)$ is defined, besides the constraints $[p, c^+]$, by the inequalities

$$z_A \leq \widetilde{v}^+(A) = v^+(A \cup (S \cap M_+)) - x_{S \cap M_+} \text{ for all } A \subset M_+ \diagdown S$$

and the equality $z_{M_+ \diagdown S} = x_{M_+ \diagdown S}$. Thus $\mathcal{C}(R)$ is the core$^+$ of the concave game $(M_+ \diagdown S, \widetilde{v}^+)$ and the proof proceeds exactly as in step 1. We omit the details.

We have proved that $\mathcal{H}^+(m-1)$ holds for any choice of $\widetilde{R}$. Now consider an arbitrary profile R such that $M_+(R)$ is not empty. Choose i in $M_+(R)$ and such that $p[R_i] \leq p[R_j]$ for all j in $M_+(R)$. We can clearly construct a profile $\widetilde{R}$ of mutually constrained-equal preferences such that $\widetilde{R}_i = R_i$ and $p[\widetilde{R}_j] \leq p[R_j]$ for all j in $M_+(R)$. Then $R \in \mathcal{D}(\widetilde{R}, m-1)$ and the proof that ψ and ψ^e coincide on $M_+(R)$ is complete.

Step 2b shows that ψ and ψ^e coincide on $M_-(R)$ by a symmetrical argument. It is omitted for brevity.■

7 Concluding comments

1) *Summary of results* Our model generalizes to a considerable extent the standard division model introduced by Sprumont (1991). This extension generates several hurdles because of additional feasibility constraints imposed by the bipartite graph. The division of the graph in three submarkets in which there is over demand, balancedness, and under demand respectively, gives the structure of the Pareto optimal allocations. Then the feasibility constraints are captured by a system of submodular upper bounds on coalitional shares in the underdemanded segment of the market, and a system of supermodular lower bounds in the underdemanded segment. Finally equals.equal treatment of equals must be restricted to those equalizing transfers that do not affect the shares of agents not involved in the transfer. After those new features are properly incorporated,

our Egalitarian solution is still characterized by the combination of efficiency, strategyproofness and equal treatment of equals.

2) The Uniform solution in the standard model satisfies many other desirable properties, and some of these are easily adapted to our model. Examples include monotonicity with respect to to changes in the population, the resources or the preferences. Consider for instance

Resource monotonicity: *A rule ψ is resource monotonic if for each $R \in \mathcal{R}^M$ and each $\omega' \geq \omega$, $\psi_i(R, \omega) \leq \psi_i(R, \omega')$ for each $i \in M$.*

Clearly this property –appearing in Moulin (1999)– is satisfied by the Egalitarian solution. This is also the case of Population monotonicity and Welfare domination under preference replacement, a formal definition of which we omit for brevity (see Thomson (2005)).

Another property playing a central role in characterizing the parametric rules (including the uniform rule and much more) is Consistency (Young (1985)). It is adapted to our model as follows. Consider a solution ψ defined for all problems $(G(M, Q), \omega, c, R)$, (in particular for problems involving smaller sets of agents), and pick an arbitrary $G(M, Q)$-flow φ implementing $\psi(G(M, Q), \omega, c, R)$. We assume agent i leaves the problem with his share of φ, resulting in a flow $\varphi(-i)$ where all edges between i and Q are deleted, and the resources correspondingly reduced to $\omega(-i)$: Consistency of ψ requires that $\varphi(-i)$ implements the solution $\psi(G(M \diagdown i, Q), \omega(-i), c_{-i}, R_{-i})$. It is again clear that the Egalitarian solution is consistent in this sense.

3) Our model also generates possibilities for new types of monotonicity and cross-monotonicity properties that do not apply to the standard model. We discuss two such properties and illustrate them with examples.

Link monotonicity I: *A rule ψ is link monotonic I if for each $R \in \mathcal{R}^M$, each $i \in M$ and each G' such that $f_G(i) \subset f_{G'}(i)$ and $f_G(j) = f_{G'}(j)$ for each $j \neq i$, we have either $\psi_j(R, G)\, R_j\, \psi_j(R, G')$ or $\psi_j(R, G')\, R_j\, \psi_j(R, G)$ for each $j \neq i$.*

This is a solidarity property. Because no one other than i is responsible for the additional links of i, we would like each agent $j \neq i$ to be affected in the same direction. We suggest a second property which pertains to changes in links as in link monotonicity.

Link monotonicity II: *A rule ψ is link monotonic II if for each $R \in \mathcal{R}^M$, each $i \in M$ and each G' such that $f_G(i) \subset f_{G'}(i)$ and $f_G(j) = f_{G'}(j)$ for each $j \neq i$, we have (i) $i \in M_+(p, G) \implies \psi_i(R, G)\, R_i\, \psi_i(R, G')$, or (ii) $i \in M_-(p, G) \implies \psi_i(R, G')\, R_i\, \psi_i(R, G)$.*

Link monotonicity II stipulates that if i is initially in M_+ then he cannot benefit from having a new link while the opposite should hold if he is initially in M_-. We show below that the egalitarian rule violates both versions of link monotonicity.

Example 7 *The egalitarian rule violates link monotonicity*

In the Figure below –a replication of Figure 3, there are two submarkets (M_+, Q_-) *and* (M_-, Q_+) *and the egalitarian allocation is* $(13, 15, 9, 5)$*.*

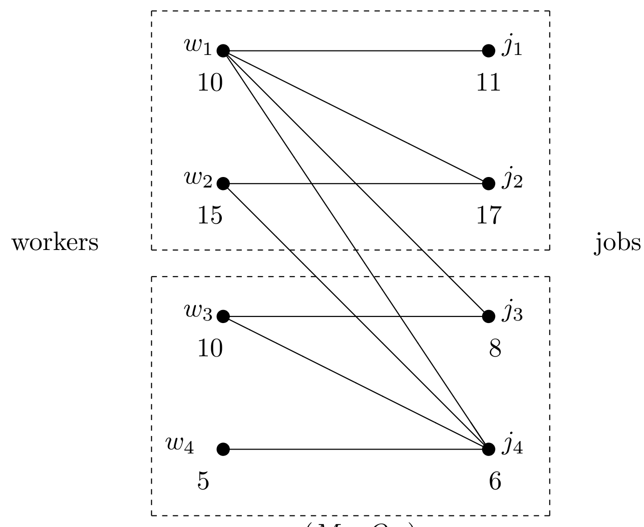

Figure 6: Initial situation

Now suppose that w_3 *gets a new link with* j_1*. The situation is depicted in the Figure below. The addition of the link changes the GE decomposition. There is only one submarket* (M_+, Q_-) *remaining. It can be checked that the egalitarian allocation is* $(10.5, 15, 10.5, 6)$*. Both link monotonicity I and II are violated. For link monotonicity I, observe that while agent* w_1 *is now closer to his peak, agent* w_4 *has been hurt by the change in the decomposition. To see that link monotonicity is violated, let simply preferences* R_{w_3} *be such that* $9 \, R_{w_3} \, 10.5$*.*

◇

4) We mention finally two possible extensions of our work. First, following Sasaki (1997), Ehlers and Klaus (2003), we can think of a "discrete" variant where indivisible units have to be distributed. Both papers above offer a characterization of the randomized Uniform rule, and it is likely that their result can be adapted to include bilateral constraints.

Next, we have has considered only symmetric rules. In the standard model the rich families of allotment rules (Barbera Jackson and Neme (1997)) preserve the incentive properties of the Uniform rule while allowing a very different treatment of the agents. Similarly the family of fixed paths rules (Moulin, 1999) is characterized by the combination of efficiency, strategyproofness, resource monotonicity and consistency. Both families can naturally be extended to our model, though the corresponding characterization results, if any, would require further research.

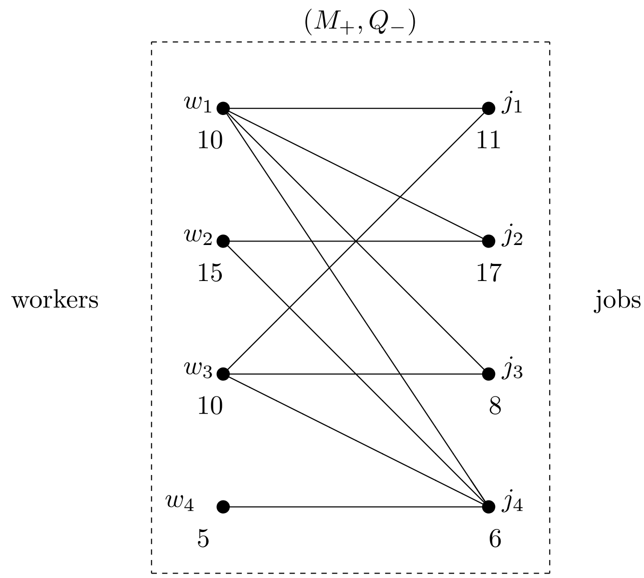

Figure 7: New situation: adding a link

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