Asymmetric internal leadership in confronted parties

Julen Berasaluce*
directed by Enriqueta Aragonès
Departament d'Economia i d'História Económica
Universitat Autónoma de Barcelona
08193 Bellaterra (Barcelona), Spain

October 7, 2009

Abstract

In this paper I provide a model of endogenous generation of leaders in a setup where two well differenced parties run for office. The unique equilibrium is characterized. Then the some of the factors that may explain a different number of leaders between parties are studied. It is examined how introducing asymmetries in the distribution of voters may be a sufficient factor to justify such difference. Not only that, but also the combination of asymmetries in the distribution of voters with a decreasing influence function or some structure inside the party can explain the different number of leaders among parties.

Key Words Endogenous leaders, voter mobilization.
JEL Classification: D72.

*thanks to Fundación Areces for the financial suppor and Hannes Mueller, Clara Ponsati, Torsten Persson, Rebecca Morton, Sjaak Hurkens, Wiola Dziuda, Maik Schneider and participants of the II Summer School in Public Economics at the IEB for their helpful comments.

1 Introduction

One of the questions that has created greatest discussion in political economics is whether the act of voting is rational. The answer to this question depends on how the act of voting itself and the consequences of such an action can affect the utility of the agent that is taking this action. However, there has been little success in explaining the action of voting whenever the weight of a single vote, and therefore, the agents' utility, is tiny.

I would like to consider the model of Riker and Ordeshook [**?**] as an initial reference when facing the preceding problem. They first argued about the equation that a rational voter should face when deciding if going to vote or abstaining. An individual will only vote if the benefits from voting, that is, what she gains if her preferred party wins the election weightened by the probability that her vote will change the outcome, is bigger than her costs of going to vote.

However, in many voting procedures, the probability that a single vote might change the result is very small. For instance, in the U.S. the probability that a single vote will affect the final outcome of a presidential election has been estimated to be around 1 in 10 million. Even if we only considered some small states in particular close elections (such as Nevada in 1960 or Alaska in 1972), this probability could only rise to 1 in 1.5 million (Gelman A, King, G and Boscarding W.J.) [**?**].

When considering voting costs and the benefits that an individual may get from voting, one needs to refer to the paradox of not voting. According to this, when there are many voters, people should never vote because one vote is never enough to change the result in a way that is relevant to the voter's utility. However, if everybody acted like this and decided not going to the polls, one vote would be enough to determine the winner.

The paradox of not voting has being approached in very different ways. Changing the essence of the vote has been one way to avoid this problem. Instead of considering the vote as a tool that may decide which party is in government, voting has been analyzed as the final objective of the action, which directly yields benefit to the voters. People use their vote as a way of expressing themselves and obtain a positive utility from that. Thus, the paradox of non voting automatically disappears. Not only that, but this hypothesis has been useful to explain many voting behaviors.

However, the instrumental approach to voting has not been forgotten. For many people, it is hard to think about voters feeling relieved, once they have fulfilled their right to vote, the action which, according to the expressive approach would be the responsible of their utility gain. Instead of that, most citizens are nervous until the results are given. Once the winner of the elections or the partition of the Parliament is revealed, nerves are calmed down.

One of the reasons that has been given to explain high participation in large elections under the instrumental approach is the overestimation of tiny probabilities. According to G. Quattrone and A. Tversky [?], voters would assign a greater weight to their vote than the one it really has. They also explain how some of the voters are affected by the so called 'voters illusion'. This effect studies how individuals decide to vote, because they are afraid that if they abstained, those who had similar preferences to them would do exactly the same. They assume that their action is having an effect on others' action, even if it has not.

A different way of facing this problem, following again the instrumental approach, is assuming that only a subset of the population rationalizes the problem stated above. Each member of this small set of people affects, through their vote and their influence on the rest of voters, with a greater probability. This asks for the rest of voters to act as, or, at least, to be influenced by, the ones taking the decision. These models which divide the citizens between leaders and followers have commonly been defined as group-based models.

The number of leaders belonging to each one of the parties is an issue of great importance in this model, because these are the ones that obtain effective votes for their respective party. For this reason, the objective of this paper is to analyze for the different features that may explain a larger number of leaders in the society, which may induce a higher participation, as well as the factors that could induce a different number of leaders between opposite parties. This difference in the number of leaders is vital, because it could sometimes be the only reason to explain a winning party.

Before analyzing the methodologies, perspectives and results this literature has provided, we should ask ourselves if belonging to any specific group does affect someone's decisions. It has been examined how membership to social groups may explain political behavior. Brooks and Manza [?] have shown how the belongship to an ethnic group and individuals' gender have, since 1960, increased their significance as explanatory variables to picture voters' behavior. The importance of the religion and the social class of individuals has been relatively stable since then. Anyway, the late expanssion of global communications do not seem to disminish the effect of these characteristics, cause and effect of the interaction of people, on voters behavior. Belonging to the same social group could imply these individuals having similar preferences and, therefore, behaving similarly. Since such a study is not able to identify if the cause of a homogenous behavior are the similar preferences or a potential influence, we should not discard any of them.

If we considered the treatment that social sciences have given to the influence from leaders to followers, we would notice that their importance is beyond the fact that group based models avoid the non voting paradox. And, even if ideological closeness is enough reason to explain similar behavior, it should be analyzed how such an important concept in sociology as it is influence, may affect the result of some elections.

There have been many attempts in order to classify the society with respect to the influence that different groups have among them. This classification has been made, in many occasions, following discrete variables. Although limiting the whole population to a discrete, or even a dichotomous choice, may not be very realistic, this is the methodology that has been used by many sociologists. There has been a tendency not to stratify too much the society with respect to personal influence, which allows for a good description of each group.

A good example of that is given by Elmo Roper [**?**] who divides, theoretically, the society between 6 levels of influence and gives a rough estimation about how many US citizens would be in each level. According to him these would be the six levels from the most important in power to the less important one, although only the last two of them, the Participating Citizens (10 to 25 million) and the Politically Inert group (75 million) include a significant part of the population, large enough not to be considered as elected politicians nor negligible among the whole population (the number given in parentheses corresponds to the estimations of people in each group of US citizens in 1954).

The division of the general public between Participating Citizens and the Politically Inert group is quite consistent with a large literature dealing with the two-step flow hypothesis. This was started by Lazarsfeld et al. (1948) [**?**] and it has in common with the preceding that it also divides the general public between two main categories: opinion givers and opinion receivers. This formulation, indeed, differs somehow from the preceding one, because it assumes that opinion givers influence on opinion receivers, while from the perspective of Roper the last group was not participating in any political discussion.

This literature was criticized because of its extreme simplicity, especially when the media developed so that a slightly more complex structure was needed to examine the phenomena. The critique on which this paper is partially based is the already mentioned study by Robinson [?] (1976). He argues about the different kind of people that are considered in the group of opinion receivers, especially after the huge development of media at that time. He made the hypothesis of differentiating two kinds of people among those that are opinion receivers: the ones that only attended to media and those who looked for a more personal source of opinion. The latter ones are the ones that we are interested in. They looked for a source of opinion closer to their own one than the one they would receive from the media. Although he did not provide a formal model to explain these effects, he proved the existence of these groups and the flows of information with a survey.

There have also been group based models in Political Economy, which must not be confused with the citizen candidate models. The basic difference of the group based models from the rest of the models which also have an instrumental approach is that the first ones move the strategic cost-benefit analysis one step up. Leaders are agents created by political parties who will get a large benefit from the victory of their party. So, they incur in large costs to convince other citizens to vote for this party. The private benefits of leaders should be defined, so that a group based model is not considered a citizen candidate model. Citizen candidate models (Besley and Coate [?]) assume that leaders have political objectives. According to that, they are the ones running for office and, therefore, voters care about the features of the candidates, instead of the parties'. This is not the case in group based models where leaders are trying to get the votes, not for themselves, but for the party.

Throughout the description of the model, given in section 2, it can be noticed that a particular leader may only influence a follower if they agree on the candidate. Assuming that leaders cannot switch followers opinion, but convince them to take an action they are satisfied with is a common feature of these models (Sachar and Nalebuff [?]). This reflects that it is much easier for leaders to mobilize votes than to convince those who think differently.

Unlike many of the models, which take exogenous leaders (Sachar and Nalebuff, [?]; Glaeser et al. [?]), the number of leaders in this paper is endogenous. These leaders could be taken as the directors of endogenously formed groups. A condition to establish a group is that its members must have similar views. The creation of these groups will be based on a concept called homophily. Homophily is the natural tendency of individuals to associate with those who think similar to them. This phenomena was analyzed by Joseph A. Precker [?], who, in a study, found that most individuals like others who are similar to them, even before any mutual friendship has been stated. Lazarsfeld and Merton [?] distinguished between status homophily and value homophily. Value homophily is the tendency that individuals have to join those who think similar to them. On the other hand, status homphily means that people who are in the same social status tend to associate among them. However, these two concepts can be correlated when we talk about political views. For the purpose of this analysis, I will only focus on value homophily.

Herrera and Martinelli [?] examined a model, where leaders endogenously arised. They described the equilibrium and some of its features in a circular space. However, some of the essence of the problem of the political competition cannot be analyzed in a circular space. If we consider the traditional left-right division, the two extremes have little in common except their extremism (in any case, much less in common than two centrist voters). The construction of the model in this paper has some elements in common with the one of Herrera and Martinelli [?] and he main difference is that we move from the circular space to a one dimensional one. This transition needs some different treatments and allows for new features to be considered.

We need to say a few words about how influence works in this paper. Influence has been modelled in very different ways. It has been treated as a channel in order to share information. This approach has been taken by A. Mattozzi [?]. Similar to this paper, there are other authors who have explored possible equilibria in models with influence, where this effect allows some individuals to have control over some others. Herrera and Martinelli [?], and Glaeser et al. [?] are good examples of models where followers act exactly in the same way as the leaders who are influencing them.

In this paper, a leader influences, and so convinces to vote, to all those voters that have an ideal in the political segment between him and the next most extremist leader of that party. The idea of why all those individuals would be influenced by that leader is not just a convention. This assumption is similar to the long term equilibrium condition that Glaeser et al. [?] got for discrete ordered populations, when examining criminal behavior. In their model, which can be easily applied to political economy[1], leaders were individuals who would not get influenced, while the rest are individuals who would imitate the behavior of those standing next. The way influence works in our continuum is equivalent to the Glaeser et al.'s [?] long term equilibrium, whenever the indivduals standing at the extremes acted as leaders.

The model I discuss is an adaptation of the one by Martinelli and Herrera [?] from the circle to the one dimension. This is closer to the reality of

[1] They assume that criminality is an issue that could be transmitted due to the closeness among individuals in some sociological variables, depicted in a one dimensional line. The proximity among individuals at the ideological space, due to which, influence must be greater, has been treated exactly the same.

parties in most of the political competitions. Not only that, but the location of any agent along the line has now a known meaning that might be useful to apply to reality, that is, ideal policies. I will check how the main features of the equilibria are equivalent in this setup as well. Leaders and followers will arise endogenously while there will always exist some politically inert citizens. These, due to the construction of the model, will always be located in the center.

In section 2, the basics of the model and the definition of equilibria are given. Moreover, I precisely define the equilibrium for the continuous case where parties have voters and leaders uniformely distributed. I characterize the unique equilibrium and prove its existence. Furthermore, I examine some comparative statics in this framework. Extending the analysis to non uniform distributions is too complex and it does not yield any clear theoretical result. However, the model can be a good tool when making simulations with the different distributions. In order to give an intuition, I have examined runs where the potential voters of each party arise from different beta distributions. These results cannot be so easily generalized to any distribution, but show us the incidence that asymmetries in distributions may yield asymmetries in the number of leaders. It is also mentioned in the paper how the introduction of an influence function decreasing in the ideological distance may alter the results.

In section 3, I jump from the continuous case to a discrete one. The objective is not to analyze an election with few voters, but one where parties have an internal structure. This internal structure is characterized by some previously defined groups, each of them having a positive measure of votes[2]. By extending the leaders followers model to the internal structure of one party, we examine how asymmetric distributions combined with an internal structure may play a role explaining differences in the number of leaders. Two general cases are analyzed. In the first one, the internal structure of one party is biased to the center, while in the second one it is biased to an extreme.

A resume of the results, conclusions and ideas for further research are offered in section 4.

[2]Unlike the continuous case, where leaders' votes as well as individually considered followers' are neglictible.

2 The model. Continuous case.

Let's assume a political competition between two parties: A and B.

The ideal policies of the population are distributed in the policy space $[-1, 1]$ as follows. Half of the population is uniformely distributed over the segment $[-1, 0]$ and the other half is uniformely distributed over $[0, 1]$ [3].

There is a countable set of leaders for both parties A and B. Leaders of party A are randomly distributed over the segment $[-1, 0]$ and those of party B are randomly placed over $[0, 1]$. Leaders are the connection between parties and voters and their distribution along the policy space is the same as the voters' one.

Leaders are parties' tools. The only way parties are able to obtain votes is through leaders. Leaders represent political activists that obtain, through their work and connections, votes for their preferred party. A leader convinces to vote to some potential voters, following a very easy rule: each leader convinces to vote to all potential voters who have a more extremist ideal policy. Since leaders form a countable set, their weight in the elections is negligible and so, we will just focus on the convinced voters.

In order to explain how leaders convince voters, assume that we have L_A party A leaders and L_B party B leaders. Each of these leaders has an ideal policy which is uniformely distributed along the policy space of their respective party, exactly the same as voters. Take all the leaders of each of the parties and order them with respect to the distance from 0 to their respective ideal policy. Thus, $x_{1,A}$ is the party A's leader who has an ideal policy closest to 0 (the most centrist one) and $x_{L_A,A}$ is the party A's leader who has the ideal policy furthest from 0 (the most extremist one). Then, being $x_{i,A}$ the ideal policy of some leader i, she will convince to vote to all those voters who have an ideal policy in the segment $(x_{i-1,A}, x_{i.A}]$.

The description given above in order to explain how party A leaders obtain votes for their party works exactly the same for party B (see figure 1, where each arrow starts from the ideology of the leader and covers all the space of those who are being influenced).

The number of leaders is decided by parties, which are office motivated. Getting one extra leader implies a fixed cost for the party equal to C. Winning the elections implies a benefit, for the moment, equal for both parties.

[3]The population could be obviously distributed uniformely $[-1, 1]$. These distributions are given separately because modifications are done separately as well.

However, since, by the construction of the model, the parameter of interest to parties is the ratio cost-benefit, the benefit of winning the elections can be normalized to 1. Thus, C is understood as the cost of acquiring one leader, when parties enjoy a benefit equal to 1 for winning the elections or consider that C is the ratio "cost of a leader" over "benefit of winning".

When deciding how many leaders parties want, they have to compare the cost of having one extra leader, C, with the extra probability that they would obtain with that leader. If the difference in the probability of winning the election is greater than the cost, then parties want more leaders; otherwise they do not.

In order to characterize the equilibrium, a few definitions are needed first. Let L_A and L_B be, respectively, the number of leaders of party A and B. Then, $P_j(L_A, L_B) \geq 0$ for $j = A, B$ is the probability that party j wins the elections. Since there are only two parties it is obviously the case that $P_A(L_A, L_B) = 1 - P_B(L_A, L_B)$, that is, the sum of both probabilities must be one.

Moreover, we will define the marginal benefit. The marginal benefit is the increase in the probability of winning carried by the last leader of that party. That is, the difference in the probability of winning compared to the situation where the party had one leader less. Name $MB_j(L_A, L_B)$ the marginal benefit of party j, then:

$$MB_A(L_A, L_B) = P_A(L_A, L_B) - P_A(L_A - 1, L_B)$$
$$MB_B(L_A, L_B) = P_B(L_A, L_B) - P_B(L_A, L_B - 1)$$

Then, an equilibrium should be a pair (L_A, L_B) such that adding a new leader would not carry an increase in expected benefit large enough compared to the cost of having a new leader. Equivalently, having one less leader would imply a reduction in cost, compared to the loss suffered in expected benefit. This conditions are summarized in the following definition of equilibrium.

Definition 1. *A pair (L_A, L_B) is an equilibrium as long as:*

$$MB_A(L_A, L_B) \geq C \text{ and } MB_(L_A + 1, L_B) < C$$
$$MB_B(L_A, L_B) \geq C \text{ and } MB_(L_A, L_B + 1) < C$$

2.1 The equilibrium with symmetric uniform distributions.

I will consider the features given in the description of the basic model as the benchmark. In this section we characterize some of the equilibrium conditions of this setup. In the following sections we change some of the hypothesis to analyze how the results obtained here change.

Consider the setup given above and take L_A and L_B leaders for parties A and B respectively, so that we have totally $L_A + L_B$ leaders. Take each one of the parties $j = A, B$ separately and put in order its leaders according to the following rule: for each party $j = A, B$ we will name $x_{1,j}$ the ideal policy of the leader of party j who is nearer from 0; $x_{2,j}$ will be the second one and we will follow this rule so that $x_{L_j,j}$ is the ideal policy of the leader who is the furthest from 0, that is, the most extremist .According to this rule, if we represent the ideals of the leaders from the smallest to the biggest whe have the following serie:

$$x_{L_A,A}, x_{L_A-1,A}, ..., x_{1,A}, x_{1,B}, x_{2,B}, ..., x_{L_B,B}$$

We will use order statistics of random variables, because leaders are randomly taken from the distribution of the population. It will be useful to remember that the denstity function of the kth order statistic ($f_{x_k}(x)$) as a function of the cumulative distribution function and the density function of a random variable is given by:

$$f_{x_k}(x) = \frac{n!}{(k-1)!(n-k)!} F(x)^{k-1} [1 - F(x)]^{n-k} f(x)$$

where $F(x)$ and $f(x)$ are, respectively the cumulative distribution function and the density function of x.

Note that when computing the number of votes that go to each of the parties, we have only two important elements: $x_{1,A}$ and $x_{1,B}$. The location of the ideal policies of the most centrist leaders of each party are going to determine the total number of votes that go to each of the parties. However, due to the randomization of their location the total number of leaders is going to be the variable of importance.

Divide all the leaders in two separate groups according to their preferred party and maintain them ordered with respect to their ideal policies, from the smallest (closest to -1) to the largest (closest to 1). Thus, we will have two groups: the ideal policies of the leaders of party A ($(x_{L_A,A}, x_{L_A-1,A}, ...x_{2,A}, x_{1,A})$ and those of B $(x_{1,B}, x_{2,B}, ..., x_{L_B-1,L_B}, x_{L_B,B})$. Take, for each of the two parties, the ideal policy of the leader who is the nearest from 0, these are $x_{1,A}$ and $x_{1,B}$. These two variables are order statistics and we will calculate their density function.

Let's start with the density function of $x_{1,B}$, which is the first element of a group of L_B that are taken from a uniform distribution. Then its density function is given by $f_{x_{1,B}}(x_B) = L_B(1 - x_B)^{L_B-1}$, defined over the segment $[0, 1]$.

On the other hand, $x_{1,A}$ is the last element of the grup of L_B leaders of party B, where all of them are uniformely distributed. Then, the density function of $x_{1,A}$ is given by $f_{x_{1,A}}(x_{1,A}) = L_A(x_{1,A})^{L_A-1}$. In order to make the computations easier we will use a transformed variable $z_{1,A} = 1 + x_{1,A}$. This transformation makes the calculus due to two reasons: first,$z_{1,A}$ has the same density function as $x_{1,A}$ ($f_{x_{1,A}}(x_{1,A}) = L_A(x_{1,A})^{L_A-1}$) and, second, $z_{1,A}$ is defined in the segment $[0, 1]$, the one over we have defined the density function of $x_{1,B}$

Since these two random variables are independent, we can easily define their joint density function:

$$f_{z_{1,A},x_{1,B}}(z_{1,A}, x_{1,B}) = L_A L_B (z_{1,A})^{L_A-1}(1 - x_{1,B})^{L_B-1}$$

Now, we will analyze the probability of winning of party A. This is the probability that the length of the segment $[-1, x_{1,A}]$ is greater than the one of $[x_{1,B}, 1]$. That is, the probability that $[x_{1,A} - (-1)]$ is greater than $[1 - x_{1,B}]$. Or, equivalently, the probability that $z_{1,A}$ is greater than $[1 - x_{1,B}]$. Therefore:

$$P_A(L_A, L_B) = \int_0^1 \int_{1-x_{1,B}}^1 L_A L_B z_{1,A}^{L_A-1}(1 - x_{1,B})^{L_B-1} dz_{1,A} dx_{1,B}$$

$$= \int_0^1 L_B(1 - x_{1,B})^{L_B-1} dx_{1,B} - \int_0^1 L_B(1 - x_{1,B})^{L_A+L_B-1}$$

$$= -1(-1) + \frac{L_B}{L_A + L_B}(-1) = 1 - \frac{L_B}{L_A + L_B} = \frac{L_A}{L_A + L_B}$$

Then, it is obviously the case that $P_B(L_A, L_B) = \frac{L_B}{L_A+L_B}$.

Now, that we have $P_j(L_A, L_B)$ for $j = A, B$, we can compute $MB_j(L_A, L_B)$.

$$MB_A(L_A, L_B) = P(L_A, L_B) - P(L_A - 1, L_B) = \frac{L_A}{L_A + L_B} - \frac{L_A - 1}{L_A + L_B - 1}$$

$$MB_A(L_A, L_B) = \frac{L_B}{(L_A + L_B)(L_A + L_B - 1)}$$

Lemma 1. *If the ideal points of the voters of each of the parties are uniformely distributed and the area of influence of each party have equal size, then the marginal benefit of a party decreases with the number of leaders of that party (i.e.* $\frac{\partial MB_j}{\partial L_j} < 0$ *for* $j = A, B$*).*

All proofs can be found in the Appendix.

Lemma 2. *If the ideal points of the voters of each of the parties are uniformely distributed over areas of influence of equal size, then the marginal benefit of the party with more leaders increases with the number of leaders of the oppsite parties (if* $L_j \geq L_k$*, then* $\frac{\partial MB_j}{\partial L_k} \geq 0$*).*

See Apppendix

The intuition of the Lemma 2 is the following. Assume we are analyzing the decision of party A and, in that situation, it has many more leaders than party B. Then, even if it is quite sure that party A is going to win the election, it would win the election anyway without one of those leaders. However, if the number of leaders in party B is bigger and bigger, so that it comes closer to the number of leaders in party A, losing one leader becomes a more important issue for party A.

2.1.1 Existence and uniqueness of equilibrium.

Since we want to completely characterize the equilibrium, we should first notice that the probabilities and marginal benefits we have calculated up to this point are for positive values of L_A and L_B greater than 1. Then, let's analyze all the possible cases.

First of all, may $(0, 0)$ be an equilibrium? In order to be so it is necessary that deviations would not be optimal. Going from such situation to one where one of the parties had only one leader, would imply improving the probability of winning from $\frac{1}{2}$ (which is the probability of winning for each party when there are no leaders) to 1 for the party that has one leader. When there are no leaders, nobody would vote, and since both parties would draw, each of them would win with probability $\frac{1}{2}$. Then, the $MB_A(0, 0) = MB_B(0, 0) = \frac{1}{2}$. Hence $(0, 0)$ is an equilibrium as long as $\frac{1}{2} < C$.

Lemma 3. *A pair of strategies* $(L + j, L)$ *for* $j > 0$ *can never be an equilibrium.*

See Apppendix

So, if the equilibrium exists it must be the case that is of the form $(L^\star, L^\star)$. In that case, the marginal benefit for each party would be $MB_j(L^\star, L^\star) = \frac{1}{2(2L^\star-1)}$. The marginal benefit is strictly decreasing in L, it goes to zero when L goes to infinity and when $L = 1$, $MB_j = \frac{1}{2}$. Having described this features note that there will always be an equilibrium. It will be $(0, 0)$ if $C > \frac{1}{2}$. Otherwise take the biggest L that makes $\frac{L}{2L(2L-1)} \geq C$ hold. That will be an equilibrium because the marginal benefit is strictly decreasing and tends to zero.

Proposition 1. *If the ideal points of the voters of each of the parties are uniformely distributed and both areas of influence are of equal size, then the equilibrium exists and is unique. If $C > \frac{1}{2}$, then the equilibrium is $(0, 0)$. On the other side, if $C \leq \frac{1}{2}$, then the equilibrium is having $L^\star$ leaders in each party, where $L^\star$ is the biggest L for which the inequality $\frac{L}{2L(2L-1)} \geq C$ holds.*

2.1.2 Cost assymetries.

Let parties A and B have different costs in order to create a leader. There might be many reasons that could explain such difference in costs. There may be historical reasons that make the participation of leaders more usual in one party; the interaction of the political party with other institutions, such as the religious ones, where leaders are very common;...

Proposition 2. *If one of the parties has smaller leadership costs than the other ($C_i < C_j$), then it will have weakly more leaders than the other party ($L_i \geq L_j$).*

See Appendix

The conclusion of this proposition is very clear and intuitive. Whenever one of the parties has smaller (marginal) costs to obtain more leaders than the other party, it will have no less leaders in equilibrium.

2.2 Asymmetric distributions.

2.3 Electorates with different size.

Now, let's consider an asymmetric situation, where the B party suffers from a disadvantage. It could be that due to worse communications channels or their lower leadership skills, the leaders of one party only obtain a fraction of votes from the people they influence. We will maintain all the original assumptions, but the party B is only going to receive a fraction $\frac{1}{\beta}$ of the votes, where $\beta > 1$[4]. Then, the joint density function of $(z_{1,A}, x_{1,B})$ would not change. But the probability for party A winning the elections would be equal to the probability of having $z_{1,A} \geq \frac{1-x_{1,B}}{\beta}$. We could calculate that using the following integral:

$$\int_0^1 \int_{\frac{1-x_{1,B}}{\beta}}^1 L_A L_B z_{1,A}^{L_A-1}(1-x_{1,B})^{L_B-1} dz_{1,A} dx_{1,B}$$

$$= \int_0^1 L_B(1-x_{1,B})^{L_B-1} dx_{1,B} - \int_0^1 \frac{L_B(1-x_{1,B})^{L_A+L_B-1}}{\beta^{L_A}} dx_{1,B} = 1 - \frac{L_B}{(L_A+L_B)\beta^{L_A}}$$

From here, we can easily compute the decissiveness of party A:

$$MB_A(L_A, L_B) = 1 - \frac{L_B}{(L_A+L_B)\beta^{L_A}} - 1 + \frac{L_B}{(L_A+L_B-1)\beta^{L_A-1}}$$

$$= \frac{L_B[1+(\beta-1)L_A+(\beta-1)L_B]}{(L_A+L_B)(L_A+L_B-1)\beta^{L_A}}$$

As well as the one for party B:

$$MB_B(L_A, L_B) = \frac{L_B}{(L_A+L_B)\beta^{L_A}} - \frac{L_B-1}{(L_A+L_B-1)\beta^{L_A}}$$

$$= \frac{L_A}{(L_A+L_B)(L_A+L_B-1)\beta^{L_A}}$$

This tools are, indeed, very useful to justify the following proposition, which could be named the 'crushing' effect:

Proposition 3. *If leaders from party i obtain a higher proportion of votes from their area of influence than those of party j, then party i would have weakly more leaders than party j ($L_i \geq L_j$).*

See Appendix

It's easy to see that whenever the leaders of one party are more efficient due to, for instance, better communication, these leaders are going to have a greater marginal benefit. Therefore, whenever we are in a situation when both parties have equal number of leaders, the party with the more efficient leaders is having a greater incentive to obtain additional leaders compared to the other party. This phenomenon, combined with lemma 2, makes impossible the existence of any equlibrium where the party with the most efficient leaders has less of them in equilibrium.

[4]This would be equivalent to assume that party A has a larger electorade

Corollary 1. *If leaders from party i obtain a higher proportion of votes from their area of influence than those of party j, then party i is expected to win the elections.*

In order to see this, calculate the expected winning margin in favour of the A party. One way to do it is by calculating:

$$\int_0^1 \int_0^1 (x_A - \frac{x_B}{\beta}) L_A L_B x_A^{L_A - 1} x_B^{L_B - 1} dx_B dx_A$$

$$= \frac{\beta L_A(L_B+1) - L_B(L_A+1)}{(L_A+1)(L_B+1)} = \frac{L_A L_B(\beta - 1) + L_A\beta - L_B}{(L_A+1)(L_B+1)}$$

which is strictly greater than zero when $L_A \geq L_B$.

2.4 Unequal length uniform distributions

In the followign subsection assume that the ideals of the potential voters of party B are uniformely distributed over the segment $[0, \alpha]$, where $\alpha > 1$ and everything else remains as before. In order to keep the equal size of the parties we must redefine the density function of $x_{1,B}$ and the joint density function of $(z_{1,A}, x_{1,B})$, which are respectively:

$$f_{x_{1,B}}(x_{1,B} = \frac{L_B}{\alpha}(1 - \frac{x_{1,B}}{\alpha})^{L_B - 1}$$

$$f_{z_{1,A},x_{1,B}}(z_{1,A}, x_{1,B}) = \frac{L_A L_B z_A^{L_A - 1}}{\alpha}(1 - \frac{x_B}{\alpha})^{L_B - 1}$$

The first thing we will do is calculating the probability of winning of party A. This is given by the probability of $z_{1,A}$ being greater than $\frac{\alpha - x_{1,B}}{\alpha}$, which is given by the following integral:

$$P_A(L_A, L_B) = \int_0^\alpha \int_{\frac{\alpha - x_{1,B}}{\alpha}}^1 \frac{L_A L_B z_{1,A}^{L_A - 1}}{\alpha}(1 - \frac{x_{1,B}}{\alpha})^{L_B - 1} dz_{1,A} dx_{1,B}$$

Through an easy computation of the preceeding integral (**see Appendix**) we obtain that $P_A(L_A, L_B) = \frac{L_A}{L_A + L_B}$. Since the probability of winning for party A is exactly the same as in the previous section, lemmas 1, 2, 3 and proposition 1 hold as exactly as before.

2.5 Asymmetric continuous distributions.

Due to the complex form of the order statistic's density function I have failed to extend the preceding theoretical conclusions to the general case where leaders and voters arise from general and potentially asymmetric distributions. However, we could not leave aside the asymmetric analysis as it was one of the very first objectives when we separated the parties' platforms: assuming different distributions of potential voters for each of the two parties.

Extending the calculus made in the first part for uniform distributions was a hard task even for computer calculation which leads to no sufficiently simplified result to be tractable. Instead of that, I have been calculating the expected most centrist leader for a variety of beta distributions and, after this calculating the vote share obtained by this leader. Even if this approach can yield some intuition of the problem, it only was valid to compute the expected marginal benefit of the first and the second leaders. The expected location of a third leader was unsolvable for the moment.

I have considered a distribution $Beta(\alpha, \beta)$ for one party and I have examined how the votes obtained from the expected most centrist leader change with the different values of α and β. First of all, we should consider what happens when one party decides to run with one leader (see figure 2). Then, the expected location of this leader is obviously given by the mean of the distribution. The mean of a $Beta(\alpha, \beta)$ distribution is $\frac{\alpha}{\alpha+\beta}$. Therefore we should compare the mean with the median of the population to discover which values of α and β yield the greater vote share.

Whenever $\alpha = \beta$, the $Beta(\alpha, \beta)$ distribution is symmetric with respect to its mean. Therefore the value of the median is exactly that of the mean, and, according to this model, the vote share obtained by this expected unique leader would be half of the potential voters of the party. Therefore, as long as the two parties have, respectively, symmetric populations, the vote share of the expected first leader is the same. Not only that, but since the size of the platform is equal for both sides ($[-1, 0]$ and $[0, 1]$) the distance from the ideal policy of this expected first leader to the center will be the same for both parties. Hence, even if we considered influence toward centrist followers this result would not change.

This is not the case when $\alpha \neq \beta$, because the location of the median changes with respect to the one of the mean. If $\alpha > \beta$ (resp. $\alpha < \beta$), then the mean is smaller (resp. greater) than the median and, therefore, the vote share obtained by the expected leader is greater (smaller) than $\frac{1}{2}$. Hence, if the potential voters of one party follows a beta distribution where

$\alpha > \beta$, while the one of the other party follows a beta distribution where $\alpha \leq \beta$, then the first party obtain a greater (marginal) benefit from its first expected leader. Moreover, if one of the parameters is equal for both distributions the party with the greater difference $\alpha - \beta$, is the one enjoying the greater vote share from its expected first leader. In any of the two cases, this advantaged party is obviously ready to obtain its first leader for greater leadership costs (C).

We have also seen the marginal vote share obtain by the expected most centrist leader, when assuming two leaders (see figure 3) in a population that comes from the distribution $Beta(\alpha, \beta)$. The ideal point of this leader is:

$$E[x_{1,j}(L_j = 2)] = \frac{2}{B(\alpha,\beta)} \int_0^1 x^{\alpha}(1-x)^{\beta-1} B_x(\alpha,\beta) dx =$$

$$= \frac{2\alpha B(\alpha,\beta)}{\alpha+\beta} - \frac{2\Gamma(1+2\alpha)\Gamma(\alpha+\beta)_3F_2(\alpha, 1+2\alpha, 1-\beta; 1+\alpha, 1+2\alpha+\beta; 1)}{\Gamma(1+\alpha)\Gamma(1+2\alpha+\beta)}$$

If we compute the marginal vote share obtained by the expected second leader, we can arrive to more conclusions. First of all, having $\alpha = \beta$ no longer ensures the same expected marginal benefit for the second leader. As we have said, in this case the distributions of the populations of each party are symmetric. While these parameters increase, the population of each of the parties becomes more concentrated near the mean. On the other side, if the parameters decrease, there is a higher fraction of the population near the center (i.e. 0) and the respective extreme. Note that due to the influence function assumed, parties only fail to obtain that fraction of centrist voters. Consequently, if $\alpha = \beta$ for both parties, but these parameters are greater (resp. smaller) for one of them, it obtain a higher (resp. lower) marginal benefit with its expected second leader. Therefore, the party with the greatest (resp. smallest) parameters would have more (less) leaders if the cost is low enough so that parties have more than one leader. The advantage may turn in favor of the party with the smallest parameters if the leaders also convinced the centrist followers. The obvious explanation behind this fact is that the expected most centrist leader is nearer from 0 in a distribution with smaller parameters.

But what if $\alpha \neq \beta$? Even if we cannot generally compare the marginal benefit obtained with the expected second leader for any two different distributions we can extract some conclusions. For any level of β (resp. α) examined, the marginal benefit of the expected second leader decreases (resp. increases) with *alpha* (resp. β). Then, if the two parties have a distribution of potential voters with the same α (resp. β), then the one with the greatest β (α), obtains a higher (resp. lower) marginal benefit from its second expected leader. The effect of β seems to be greater that the one of α, which would be consistent with the conclusions extracted from the case where $\alpha = \beta$.

2.6 The effect of a decreasing influence function.

Up to now we have been assuming that each leader i of party A (resp. B) would convince all those individuals with an ideal between the one of the leader and the next most extremist leaders; that is, all those having an ideal in the segment $(x_{i+1,A}, x_{i,A}]$ (resp. $[x_{i,B}, x_{i+1,B})$). The influence that a leader voting for party j with ideal $x_{i,j}$ has over a citizen with an ideal x is named $\theta(x_{i,j}, x) = \max\{0, 1-\lambda|x_{i+1,j} - x_{i,j}|\}$, where $\lambda > 0$ is a parameter that measures how influence decreases with distance. θ is assumed to be nonnegative, because we have no role for negative influence.

Dealing with general distribution functions and a decreasing influence function makes the problem much more difficult, because we are adding as many discontinuities as leaders. However, we can give the intuition for the general uniform case where the ideals of the individuals ready to vote for one party follow a distribution $U(0, \alpha)$. We are interested into calculating the expected distance between the ideals of two consecutive leaders, $(x_{i+1,j} - x_{i,j})$; that is, the expected area of influence for leader i of party j. The joint density function of $(x_{i,j}, x_{i+1,j})$ is given by:

$$f(x_{i,j}, x_{i+1,j}) = \frac{L_j!(\frac{x_{i,j}}{\alpha})^{i-1}(1 - \frac{x_{i+1,j}}{\alpha})^{L_j-i-1}}{\alpha^2(i-1)!(L_j - i - 1)!}$$

If we compute the expected area of influence, $E[|x_{i+1,j} - x_{i,j}|] = \frac{\alpha}{L_j+1}$ (**see Appendix**), we observe that it is increasing in α and decreasing in the number of leaders. Since the influence of each leader is decreasing in the distance, the party with the higher expected areas of influence will be less efficient obtaining votes. Therefore, the party which has the most concentrated population (smaller α) will be the most efficient under a decreasing influence function. Thus, such party will have no less leaders than its competitor.

Remember how we have seen that the length of the uniform distribution did not have any effect on the total number of leaders of any of the parties. However when we introduce a decreasing influence function the most concentrated population increases the efficiency of influence and may have a greater number of leaders at equilibrium.

3 Discrete case: parties with an internal structure.

In the preceding section we have been working with continuous probability density functions. Now, we are going to examine a discrete case with few players (potential leaders and followers). Examining the case with three players in each party is enough to show the effect of internal structures. The three players in each party represent three groups, ideologically differenced, inside the party. In order to make comparisons easier, I have assigned the same voting weight to each one of the groups, regardless of their respective size. However,, since we are assuming different probabilities for each group having a leader, we can easily assume that this term is including both effects: the increased probability that a leader may emerge from that group and the greater size of the electorate.

From here in advance we are going to treat these groups as individuals. Because of the internal structure, there is going to be a complete coordination inside each group. It is equivalent considering that one individual in the group becomes a leader automatically obtaining the support of the group or assuming that the whole group is the leader. The way one group is going to influence the rest is the same as in the continuous case.Therefore, the exact location of the ideals of voters does not affect the result. We will only care about the fact that for each party there is one group which is closer to zero (that may influence the other two), one which is in the middle (that may only influence the most extremist one) and one which has an ideal closer to the extreme (that cannot have an influence on other groups).

The probabilities of party A groups to become a leader are equal to $\frac{1}{3}$, when there are no leaders yet, $\frac{1}{2}$ when there is already one leader in that party and 1 if the other two groups are already leaders. For the party B groups the probabilities are different. The probability of the most centrist group becoming a leader is r; the one for the one which is the second nearer from 0 is s and the probability of the most extreme group becoming a leader is t. For the moment, we will only assume that $r + s + t = 1$. If there are only two leaders to be elected the probability of chosing one of them is equal to its original probability (say r, s or t) over the sum of probabilities of the voters who are not leaders yet. That is, if the first leader of B is the most extremist one (the one that had probability t), the probability that the second is the most centrist one is $\frac{r}{r+s}$ [5]. In the following subsections we will make different assumptions about these parameters to see which are the possible equilibrium.

[5]This will be used in order to compute ex-ante probabilities, because there is no way to know if the first leader is the most centrist one or not.

For the general values $\{r, s, t\}$, in the following table we have the probabilities of winning for party B. In the rows we have the number of leaders of party A, while those of party B are in the columns.

	0	1	2	3
0	$\frac{1}{2}$	1	1	1
1	0	$\frac{(5r+3s+t)}{6}$	$\frac{1}{3}+\frac{r}{2}+\frac{3sr+st}{6(r+t)}+\frac{3tr+ts}{6(r+s)}$	$\frac{5}{6}$
2	0	$\frac{(4r+s)}{6}$	$\frac{4r}{6}+\frac{4sr+st}{6(r+t)}+\frac{4tr+ts}{6(r+s)}$	$\frac{4}{6}$
3	0	$\frac{r}{2}$	$\frac{r}{2}+\frac{rs}{2(r+t)}+\frac{rt}{2(r+s)}$	$\frac{1}{2}$

We can, therefore, define the marginal benefit for both parties, the same as we did in the previous section:

	Party A	Party B
$(0,0)$	-	-
$(0,1)$	-	$\frac{1}{2}$
$(0,2)$	-	0
$(0,3)$	-	0
$(1,0)$	$\frac{1}{2}$	-
$(1,1)$	$\frac{1+2s+4t}{6}$	$\frac{1+4r+2s}{6}$
$(1,2)$	$\frac{r}{6}+\frac{sr+3st}{6(r+t)}+\frac{tr+3ts}{6(r+s)}$	$\frac{1}{3}-\frac{r}{3}-\frac{2st}{6(r+t)}+\frac{2tr}{6(r+s)}$
$(1,3)$	$\frac{1}{6}$	$\frac{1-r}{2}-\frac{3sr+st}{6(r+t)}-\frac{3tr+ts}{6(r+s)}$
$(2,0)$	0	-
$(2,1)$	$\frac{1+s}{6}$	$\frac{4r+s}{6}$
$(2,2)$	$\frac{2-r}{6}-\frac{sr}{6(r+t)}-\frac{tr}{6(r+s)}$	$\frac{3sr}{6(r+t)}+\frac{4tr+ts}{6(r+s)}$
$(2,3)$	$\frac{1}{6}$	$\frac{4(1-r)}{6}-\frac{4sr+st}{6(r+t)}-\frac{4tr+ts}{6(r+s)}$
$(3,0)$	0	-
$(3,1)$	$\frac{r+s}{6}$	$\frac{r}{2}$
$(3,2)$	$\frac{1}{6}$	$\frac{rs}{2(r+t)}+\frac{rt}{2(r+s)}$
$(3,3)$	$\frac{1}{6}$	$\frac{st}{2(r+t)}+\frac{st}{2(r+s)}$

3.1 A structure biased to the center.

First of all, we will characterize which are the possible equilibria when the groups of the party B are biased to the center. We will represent this by assuming that $r > s > t$. In this situation, the most centrist group in party B has a greater probability of being leader. We will analyze which is the possible set of equilibria.

3.1.1 Same number of leaders

We will start by characterizing all the possible equilibria when both players have the same number of leaders. $(0,0)$ is a special case because the possible deviation of any of the two parties (i.e. acquiring one leader) would imply for that party to obtain the highest possible marginal probability. The probability of winning changes from a fair coin toss to a sure victory of the party that deviates. Then, the marginal probability of the party that deviates is $\frac{1}{2}$. So, whenever $C > \frac{1}{2}$ $(0,0)$ is an equilibrium, and since there are no more locations where both parties obtain such a large marginal probability, $(0,0)$ will be the only equilibria in this case.

$(1,1)$ and $(2,2)$ can be equilibrium. Note that we are not ensuring the existence for any distribution of $\{r,s,t\}$. Instead of that we are ensuring that there exist some values of $\{r,s,t\}$ and C for which these equilibrium may exist. Examples can be found in the Appendix. Note as that since $MB_A(1,1) < \frac{1}{2}$ and $MB_A(2,1) \geq \frac{1}{6}$, there are no values of C out of the segment $(\frac{1}{6}, \frac{1}{2}]$, that make $(1,1)$ an equilibrium. Similarly, since $MB_A(2,2) < \frac{2}{9}$ and $MB_A(3,2) = \frac{1}{6}$, there are no values of C out of the segment $(\frac{1}{6}, \frac{2}{9}]$, that make $(2,2)$ an equilibrium.

$(3,3)$ can obviously be an equilibrium. The only requirements are that $MB_A(3,3) = \frac{1}{6} \geq C$ and $MB_B(3,3) = \frac{st}{2(r+t)} + \frac{st}{2(r+s)} \geq C$. As long as $t > 0$, we can always find values of C suffiently small for which $(3,3)$ is an equilibrium. A necessary requirement is that $C \leq \frac{1}{6}$.

Just by looking to the possible values of C for which each of the equilibria may hold, the only ones that seem able to coexist are $(1,1)$ and $(2,2)$. However, these two equilibria cannot coexist because that would ask, among other conditions, that $MB_A(2,1) < C$ and $MB_A(2,2) \geq C$. This is impossible because $MB_A(2,1) = \frac{1+s}{6} > \frac{2-r}{6} - \frac{sr}{6(r+t)} - \frac{tr}{6(r+s)} = MB_A(2,2)$ as it is shown below:

$$\frac{1+s}{6} > \frac{2-r}{6} - \frac{sr}{6(r+t)} - \frac{tr}{6(r+s)}$$

$$s + r + \frac{sr}{r+t} + \frac{tr}{r+s} > 1$$

$$r(\frac{s}{r+t} + \frac{t}{r+s}) > t$$

Given a value for r, the combination of s and r that minimizes the term inside the parenthesis is $s = t$. So, if the preceeding inequality holds for $s = t$ it does hold for $s > t$. If $s = t$ the inequality becomes $\frac{2tr}{r+t} > t$, that is, $\frac{2r}{r+t} > 1$. This is clearly true because $r > t$. Therefore, none of the equilibria with the same number of leaders for each party can coexist.

3.1.2 More leaders for party A

We will analyze all the possible pairs of leaders where party A has more leaders than B to determine which of them can and which cannot constitute an equilibrium. $(1,0)$ cannot be an equilibrium. In order to be so it would be necessary that $MB_A(1,0) = \frac{1}{2} \geq C$ and $MB_B(1,1) = \frac{1+4r+2s}{6} < C$. Clearly, $1+4r+2s = 1+2(r+s)+2r$. Since $r > s > t$ and $r+s+t = 1$, $r+s > \frac{2}{3}$ and $r > \frac{1}{3}$. So $MB_B(1,1) > \frac{1}{2}$, which is a contradiction.

$(2,0)$ and $(3,0)$ cannot be an equilibrium, because at any of them the marginal probability for party A is 0 and $C > 0$.

$(2,1)$, $(3,1)$ and $(3,2)$ can separately be equilibria. Distributions of $\{r,s,t\}$ are provided for each of the cases in the Appendix, for which there exist values of C that make them hold as equilibria. Note that since $MB_A(2,1) < \frac{1}{4}$ and $MB_A(3,1) \geq \frac{1}{9}$, there are no values of C out of the segment $(\frac{1}{9}, \frac{1}{4}]$ that make $(2,1)$ an equilibrium. Moreover, since $MB_A(3,1) \leq \frac{1}{6}$ and $MP_A(3,2) = \frac{1}{6}$, there are no values of C strictly greater than $\frac{1}{6}$ that make $(3,1)$ or $(3,2)$ an equilibrium.

If any of the two parties has incentives to deviate the equilibria cannot hold. That is why we cannot have any of the following pairs of equilibria coexisting: $(2,1)$ and $(3,1)$ or $(3,1)$ and $(3,2)$. However, may $(2,1)$ and $(3,2)$ coexist? No, because $MB_B(2,2) = \frac{3sr}{6(r+t)} + \frac{4tr+ts}{6(r+s)} \geq \frac{3sr}{6(r+t)} + \frac{3tr}{6(r+s)} = MB_B(3,2)$. Then, if $MB_B(3,2) \geq C$ it cannot be true that $MB_B(2,2) < C$. Hence, none of the equilibra where party A has more leaders than party B can coexist.

3.1.3 More leaders for party B

With the purpose of analyzing all the possible equilibria we will end by looking for these in every pair of leaders with more leaders for party B than for A. Among these, we can find that $(0,1)$ is an equilibrium as long as $MB_B(0,1) = \frac{1}{2} \geq C$ and $MB_A(1,1) = \frac{1+2s+4t}{6} < C$ (note that $MB_B(0,2) = 0 < C$ always holds). In order to prove such existence it is sufficient to find some values of $\{r,s,t\}$ for which $MB_B(0,1) > MB_A(1,1)$. This is the case, for instance, of probability distribution $\{\frac{7}{10}, \frac{2}{10}, \frac{1}{10}\}$. Since $MB_B(0,1) = \frac{1}{2}$ and $MB_A(1,1) \geq \frac{1}{6}$, there are no values of C out of the segment $(\frac{1}{6}, \frac{1}{2}]$ that can make $(0,1)$ an equilibrium.

By the same argument given in the preceeding subsection $(0,2)$ and $(0,3)$ never are equlibria.

$(1,2)$ can be an equilibrium. The prove for that is given in the Appendix. Since $MB_A(2,2) > \frac{1}{6}$ and $MB_B(1,2) \leq \frac{2}{9}$, there are no values of C out of the segment $(\frac{1}{6}, \frac{2}{9}]$ that can make $(1,2)$ an equlibrium.

$(1,3)$ and $(2,3)$ cannot be an equilibrium. Note that $MB_A(1,3) = MB_A(2,3) = MB_A(3,3) = \frac{1}{6}$. Then if it we are in the case that $MB_A(1,3) \geq C$, then it is also true that $MB_A(2,3) = MB_A(3,3) \geq C$ and party A should have three leaders.

$(0,1)$ and $(1,2)$ cannot be equilibrium at the same time, because if $(0,1)$ is an equilibrium, then $MB_A(1,1) < C$.However, $MB_A(1,1) \geq MB_A(1,2)$. This is easy to see if we rewrite $MB_A(1,1) = \frac{1+2s+4t}{6} = r+3s+5t = \frac{r}{6} + \frac{3sr+3st}{6(r+t} + \frac{5tr+5ts}{6(r+s}$. Obviously:

$$MB_A(1,1) = \frac{r}{6} + \frac{3sr+3st}{6(r+t} + \frac{5tr+5ts}{6(r+s} \geq \frac{r}{6} + \frac{sr+3st}{6(r+t} + \frac{tr+3ts}{6(r+s}$$

$$MB_A(1,1) = \frac{1+2s+4t}{6} \geq \frac{r}{6} + \frac{3sr+3st}{6(r+t)} + \frac{3tr+3ts}{6(r+s)} \geq \frac{r}{6} + \frac{sr+3st}{6(r+t)} + \frac{tr+3ts}{6(r+s)} = MB_B(1,2)$$

Hence, if the equilibria where party B has more leaders than party A exists, is unique.

In the next table, I will summarize if the pairs of leaders can be an equilibrium.

	0	1	2	3
0	Yes	Yes	No	No
1	No	Yes	Yes	No
2	No	Yes	Yes	No
3	No	Yes	Yes	Yes

3.1.4 Uniqueness of the equilibrium

Lemma 4. *In the discrete case, when one of the parties has a uniform structure and the other one is biased to the center, if the equilibrium exists, it is unique.*

Along the previous subsections we have proved that some of the equilibria cannot coexist. We will extend now the analysis to check that there is never multiple equilibria.

$(0,0)$ is equilibrium if and only if $C > \frac{1}{2}$. Since there are no other equilibria where the marginal probability is higher than $\frac{1}{2}$, $(0,0)$ will be always unique.

There is a clear argument that can be applied to justify the impossible coecistance of many pairs of equilibria. If a pair of leaders is an equilibrium, the deviation cannot be optimal and so, two pairs of leaders that differ one from another in one leader for one of the parties cannot be equilibrium at the same time. This is the case of : $(1,1)$ and $(2,1)$; $(1,1)$ and $(0,1)$; $(1,1)$ and $(1,2)$; $(2,2)$ and $(2,1)$; $(2,2)$ and $(3,2)$; $(2,2)$ and $(1,2)$; $(3,3)$ and $(3,2)$.

We will discard many coexistances if we examine the values that C can take in each of the cases. For instance, if $(3,2)$ is an equilibrium, necessarily $C \leq \frac{1}{6}$. However, $(1,1)$, $(0,1)$ and $(1,2)$ will never constitute an equilibrium for such values of C. Therefore, $(3,2)$ cannot coexist with any of the three. Similarly, if $(3,1)$ is an equilibrium, $C \leq \frac{1}{6}$ must hold, which is not compatible with either $(1,1)$, $(2,2)$, $(0,1)$, $(1,2)$ being equilibria. Following the same justification, if $(3,3)$ is an equilibrium, $C \leq \frac{1}{6}$ again, and, as we have said this prevents $(0,1)$ or $(1,2)$ from coexisting as well with $(3,3)$.

There are a few combinations of equilibrium that need to be checked yet. These are pairs of equilibria which do not correspond to the same group (since we made those comparisons along the previous subsections) and need more complex arguments than the ones given above. We need to argue why cannot be equilibrium at the same time: $(0,1)$ and $(2,1)$; $(0,1)$ and $(2,1)$; $(2,1)$ and $(1,2)$; $(2,1)$ and $(3,3)$. All these arguments are provided in the Appendix.

After analyzed, pairwise, all the possible equilibria, we can conclude that two equilibria will never coexist and, therefore, if there exists an equilibrium is unique.

3.1.5 Existence of equilibrium

Before getting into the prove of the existence we will prove two inequalities. The first one is $MB_A(2,2) = \frac{2-r}{6} - \frac{sr}{6(r+t)} - \frac{tr}{6(r+s)} \geq \frac{1-r}{2} - \frac{3sr+st}{6(r+t)} - \frac{3tr+ts}{6(r+s)} = MB_B(1,3)$. This inequality easily simplified to $1 + \frac{2st}{r+t} > r(\frac{s}{r+t} + \frac{t}{r+s})$, which is true because:

$$1 + \frac{2st}{r+t} \geq 1 > s + t = r(\frac{s}{r} + \frac{t}{r}) > r(\frac{s}{r+t} + \frac{t}{r+s})$$

The second one is $MB_A(3,2) = \frac{1}{6} \geq \frac{4(1-r)}{6} - \frac{4sr+st}{6(r+t)} - \frac{4tr+ts}{6(r+s)} = MB_B(2,3)$. We can easily simplify this to: $4r + \frac{4sr+st}{r+t} + \frac{4tr+ts}{r+s} \geq 3$. Note that:

$$4r + \frac{4sr+st}{r+t} + \frac{4tr+ts}{r+s} \geq 3r + \frac{rs+rt}{r+t} + \frac{2sr+2st}{r+t} + \frac{st}{r+t} + \frac{3tr+2ts}{r+s}$$

And:

$$3r+\frac{rs+rt}{r+t}+\frac{2sr+2st}{r+t}+\frac{st}{r+t}+\frac{3tr+2ts}{r+s}\geq 3r+\frac{3sr+3st}{r+t}+\frac{st}{s+t}+\frac{3tr+2ts}{r+s}=3$$

Having proved these, we can state and prove the following lemma:

Lemma 5. *If* $\frac{3sr}{6(r+t)}+\frac{4tr+ts}{6(r+s)}\geq C>\frac{3sr}{6(r+t)}+\frac{3tr}{6(r+s)}$ *and* $\frac{1+s}{6}\geq C>\frac{r+s}{6}$ *the equilibrium does not exist. Otherwise, it exists.*

Proof

Follow the steps in order to get the equilibrium:

1. $(0,0)$ will be an equilibrium if $C>\frac{1}{2}$. So if $C\leq\frac{1}{2}$ consider, as a first possible deviation, $(0,1)$ (because $MB_B(0,1)>MB_A(1,0)$) and go to step 2.

2. If $(0,1)$ is an equilibrium we are done. Otherwise, the only deviation that would have sense is moving to $(1,1)$, because $MB_B(0,2)=0$ (Step 3).

3. If $(1,1)$ is an equilibrium we are done. If not, add one more leader to the party that wants so. If $MB_B(1,2)\geq MB_A(2,1)$ add one leader to the B party (Step 4). Otherwise, to the A party (Step 8).

4. We are now at $(1,2)$ which can be an equilibrium. If it is not, since $MB_A(2,2)\geq MB_B(1,3)$ (proved previously) moving to $(2,2)$ is always a good deviation for A.

5. $(2,2)$ can be an equilibrium. If not, it may be because $MB_B(2,2)\leq C$ (Go to step 8). Otherwise, since $MB_A(3,2)\geq MB_B(2,3)$ party A would directly deviate to $(3,2)$.

6. If $(3,2)$ is an equilibrium or if the optimal move for party B is going to $(3,3)$ we are done (in any of these situations MB_A does not reduce and holds greater than C). However, if $MB_B(3,2)<C$ party B would move to $(3,1)$.

7. $MB_B(3,1)\geq MB_A(3,1)$, so if $MB_A(3,1)\geq C$, $(3,1)$ is an equilibrium. Otherwise, party A would prefer to move to $(2,1)$. If $(2,1)$ is not an equilibrium is because $MB_B(2,2)\geq C$ and we would be in a cycle. This situation is characterized by:

$$\frac{3sr}{6(r+t)}+\frac{4tr+ts}{6(r+s)}\geq C>\frac{3sr}{6(r+t)}+\frac{3tr}{6(r+s)}$$

$$\frac{1}{6}\geq C>\frac{r+s}{6}$$

8. If we have reached here $MB_A(2,1) \geq C$: if we come from step 3 this is obvious; if we come from step 4 note that $MB_A(2,1) \geq MB_A(2,2)$. As we have said, $(2,1)$ may be an equilibrium. If party A wants to add one more leader, we would be in an equilibrium at $(3,1)$, $(3,2)$ or $(3,3)$, depending on the best answer by B. Note that once we know $MB_A(3,1) \geq C$, we can guarantee that at least $MB_B(3,1) \geq C$. If only party B wants to deviate we would go to $(2,2)$(Step 5)

End Proof

So, once we have defined the existence, under certain conditions, and the uniqueness of equilibria, we have a clear picture of the possible different equilibria that we may obtain when one of the parties has an internal structure which is biased to the center.

We may obtain an equilibrium where party B has more leaders, like $(0,1)$ or $(1,2)$. The intuition of this is quite obvious; the marginal benefit of party B is, in principle, greater, since it has greater chances of obtaining centrist leaders, who are more productive. $(1,3)$, $(2,3)$ cannot be an equilibrium because if party A has enough marginal benefit to maintain one or two leaders when all the voters of party B are leaders; then, party A would have enough benefit to maintain more leaders.

We may also reach to equilibria where party A has more leaders, such as $(2,1)$, $(3,1)$ or $(3,2)$; but, in any of these, party B has at least one leader. This happens when the distribution of leaders is so biased to the center that is very sure for the B party obtaining a centrist leader. The marginal probability that party A gets with its second and third leaders can be, then, greater than those of B.

We also have, as in the uniform case, equilibria where both parties have the exact number of leaders. Any of these may be the equilibrium whenever r, s and t do not differ much from each other. If this is the case, the smaller C the greater number of leaders for both parties there will be in equilibrium.

There is also a situation where the equilibria does not exist. This is completely characterized in lemma **??**. In this situation party B prefers to have 2 leaders when facing 2 leaders from party A, but not when facing 3 leaders. On the other hand, party A wants to have 2 leaders, when facing one leader from party B, but not a third one. In each of the cases that may lead to this situation, we should analyze which is the winning party. The lack of equilibrium is not a bad feature by itself. It may reflect a lack of stability. Anyway, the expected winning party could assume the larger cost in order to stabilize the situation.

3.2 An extremist biased population

Now we will analyze what happens when party B has an internal structure biased to the extreme. We will represent this by assuming that $r < s < t$. In this situation, the most estremist group in party B has a greater probability of being leader. We will analyze how the possible set of equilibrium.

3.2.1 Same number of leaders

We will start by characterizing all the possible equilibria when both players have the same number of leaders. $(0,0)$ is a special case because the possible deviation of any of the two parties (i.e. acquiring one leader) would imply for that party to obtain the highest possible marginal probability. The probability of winning changes from a fair coin toss to a sure victory of the party that deviates. Then, the marginal probability of the party that deviates is $\frac{1}{2}$. So, whenever $C > \frac{1}{2}$ $(0,0)$ is an equilibrium, and since there are no more locations where both parties can obtain such a large marginal probability, $(0,0)$ will be the only equilibria in this case.

$(1,1)$ and $(2,2)$ can be separately be equlibrium. Distributions of $\{r,s,t\}$ for which these pairs of leaders can be equilibrium are given in the Appendix. Note that since $MB_B(1,1) < \frac{1}{2}$ and $MB_A(2,1) \geq \frac{1}{6}$, there are no values of C out of the segment $(\frac{1}{6}, \frac{1}{2}]$, that make $(1,1)$ an equilibrium. Similarly, $MB_A(1,1) < \frac{1}{3}$ and $MB_A(2,1) = \frac{1}{6}$, so there are no values of C out of the segment $(\frac{1}{6}, \frac{1}{3})$, that make $(2,2)$ an equilibrium.

$(3,3)$ can be an equilibrium if $MB_A(3,3) = \frac{1}{6} \geq C$ and $MB_B(3,3) = \frac{st}{2(r+t)} + \frac{rt}{2(r+s)}$. This can be an equilibrium because, as long as the marginal probabilities are postive, there always exist values of C sufficiently small that make $(3,3)$ an equilibrium.

We have already said that when $(0,0)$ is an equilibrium, it is unique. On the other side, if $(3,3)$ is an equilibrium, $C \leq \frac{1}{6}$, which is a requirement that makes impossible $(3,3)$ being an equilibrium toguether with $(1,1)$ or $(2,2)$. However, may $(1,1)$ and $(2,2)$ coexist? No, because two of the conditions that would be needed for such coexistance are $MB_B(1,2) < C$ and $MB_B(2,2) \geq C$. However $MB_B(1,2) \geq MB_B(2,2)$ **(see in the Appendix)**.

3.2.2 More leaders for party A

Let's now examine all the possible equilibria where A party has more leaders than B. One of these that can be an equilibrium is $(1,0)$ if $MB_A(1,0) = \frac{1}{2}$

and $MB_B(1,1) = \frac{1+4r+2s}{6}$. It's clear that there are many values of $\{r,s,t\}$ which make $\frac{1+4r+2s}{6} < \frac{1}{2}$, for instance $\{\frac{1}{10}, \frac{1}{5}, \frac{7}{10}\}$. Then, any intermediate value of C would make the equilibrium hold.

$(2,1)$ may also be an equilibrium. An example of this existence is given in the Appendix.

$(2,0)$ and $(3,0)$ obviously cannot be equlibria because the marginal probability of the party A is 0 in any of them and, so, cannot be greater than C.

$(3,1)$ cannot be an equilibrium. Two of the requirements that need to hold are $MB_B(3,1) = \frac{3r}{6} \geq C$ and $MB_B(3,2) = \frac{3rs}{6(r+t)} + \frac{3rt}{6(r+s)} < C$. However:

$$\frac{3rs}{r+t} + \frac{3rt}{r+s} = 3r(\frac{s}{r+t} + \frac{t}{r+s}) \geq 3r$$

In order to prove the preceeding inequality it is sufficient to see how $\frac{s}{r+t} + \frac{t}{r+s} \geq 1$, which is straightforward if we make the calculus of the right hand side and reordenate so that it becomes $\frac{s^2+t^2+r(s+t)}{r^2+st+r(s+t)}$. Since the numerator is greater than the denominator, we are done.

$(3,2)$ cannot be an equilibrium either. If it was $MB_B(3,2) = \frac{rs}{2(r+t)} + \frac{rt}{2(r+s)} \geq C$ and $MB_B(3,3) = \frac{st}{2(r+t)} + \frac{st}{2(r+s)}$. Since $st \geq rs$ and $st \geq rt$, it is clear that $MB_B(3,3) \geq MB_B(3,2)$.

Hence, the only possible equilibria with more leaders for party A are $(1,0)$ and $(2,1)$, which cannot coexist. Why? If they did, two of the conditions that would need to be fulfilled are: $MB_B(1,1) = \frac{1+4r+2s}{6} < C$ and $MB_A(2,1) = \frac{1+s}{6} \geq C$. However, it is clear that $MB_B(1,1) \geq MB_A(2,1)$.

3.2.3 More leaders for party B

$(0,1)$ cannot be an equilibrium. If it was it would be necessary that $MB_B(0,1) = \frac{1}{2}$ and $MB_A(1,1) = \frac{1+2s+4t}{6} < C$. However, since $s+t \geq \frac{2}{3}$ and $t \geq \frac{1}{3}$, it is easy to check that $MB_A(1,1) \geq \frac{1}{2}$.

$(0,2)$ and $(0,3)$ cannot be equlibrium because in any of them the marginal probability of the party A is 0 and, therefore, it cannot be greater than C.

$(1,2)$ can be an equilibrium if $MB_A(1,2) \geq C$, $MB_B(1,2) \geq C$, $MB_A(2,2) < C$ and $MB_B(1,3) < C$. Take for instance $\{r,s,t\} = \{\frac{1}{10}, \frac{1}{5}, \frac{7}{10}\}$. That would make $\min\{MB_A(1,2), MB_B(1,2)\} > \max\{MB_A(2,2), MB_B(1,3\}$. Then, there exist values of C that make $(1,2)$ an equilibrium.

$(1,3)$ and $(2,3)$ cannot be equilibrium because in any of them the marginal probability of party A is $\frac{1}{6}$. Since $MB_A(3,3) = \frac{1}{6}$, it being at $(1,3)$ or at $(2,3)$ is good for party A, it should deviate to $(3,3)$.

So, here it is the summary table with all the possible equilibria:

	0	1	2	3
0	Yes	No	No	No
1	Yes	Yes	Yes	No
2	No	Yes	Yes	No
3	No	No	No	Yes

3.2.4 Multiplicity of equilibria

When the equilibrium exists $(1,0)$ and $(1,2)$ as well as $(2,1)$ and $(3,3)$ may coexist as equilibria. Otherwise, it is unique.

We have seven possible equilibria and we have already checked that those of the same group cannot coexist. We are going to extend this analysis to all possible pairwise comparisons.

We have already said that if $(0,0)$ is the equilibrium is unique.

As we have said before two pairs of leaders that differ one from another in one leader for one of the parties cannot be equilibrium at the same time. Then the following equilibria cannot coexist: $(1,0)$ and $(1,1)$; $(1,1)$ and $(2,1)$; $(1,1)$ and $(1,2)$; $(2,2)$ and $(2,1)$; $(2,2)$ and $(1,2)$.

From the remaining combinations of paris of leaders, some cannot coexist, this is the case of: $(1,0)$ and $(2,2)$; $(1,0)$ and $(3,3)$; $(1,2)$ and $(2,1)$; $(1,2)$ and $(3,3)$ **(see the Appendix M)**.

However we have two cases where the equilibria may be multiple. $(1,0)$ and $(1,2)$ may coexist. It happens, for instance when $\{r,s,t\}$=$\{\frac{1}{100}, \frac{1}{50}, \frac{97}{100}\}$ and for an adeccuate C. If one of the following conditions holds this coexistance will break: $5r + s + 2t(s-r) \geq t$ (which makes $MB_B(1,1) \geq MB_B(1,2)$) or if $3(s+r) \geq 2st(\frac{1}{r+t} + \frac{1}{r+s})$ (which makes $MB_B(1,1) \geq MB_A(1,2)$). If none of this conditions hold, there may be multiple equilibria for an adeccuate C.

$(2,1)$ and $(3,3)$ can coexist as well as possible equilibria if $MB_A(2,1)$, $MB_B(2,1)$, $MB_A(3,3)$, and $MB_B(3,3)$ are weakly greater than C, whereas $MB_A(3,1)$ and $MB_B(2,2)$ are strictly smaller than C. This is the case when $\min\{\frac{1}{6}, \frac{4r+s}{6}\} \geq C > \frac{r+s}{6}$ and $\frac{st}{2(r+t)} + \frac{st}{2(r+s)} \geq C > \frac{sr}{2(r+t)} + \frac{4tr+ts}{6(r+s)}$. If one of these inequalities does not hold $(2,1)$ and $(3,3)$ cannot coexist as equilibria.

3.2.5 Existence of equilibria

When $r < s < t$, there are many situations that may lead us to a lack of equilibria. In order to provide a full analysis of the game we have all the possible cases and deviations. Follow the steps in order to get the equilibrium (or the lack of it):

1. Start from $(0,0)$. If it is an equilibrium, we are done, otherwise, since $MB_A(1,0) \geq MB_B(0,1)$ analyze $(1,0)$.

2. If $(1,0)$ is an equilibrium we have finished. Otherwise, the only reason is that $MB_B(1,1) \geq C$, so we will move to $(1,1)$.

3. If $(1,1)$ is an equilibrium we are done. If not, since $MB_A(1,1) \geq MB_B(1,1)$, the only reason is that any of the parties wants to obtain more leaders. If $MB_A(2,1) \geq MB_B(1,2)$ go to step 4. Otherwise, go to step 8.

4. If $(2,1)$ is an equilibrium we are done. If not, go to step 5.

5. If $\frac{1}{6} \geq C$, then $(3,3)$ is an equilibrium, so move there. If not, go to step 6.

6. Chek if $MB_B(2,2) \geq C$. If it is so, go to step 7. If $MB_B(1,2) \geq C$ $(1,2)$ is an equilibrium. Otherwise, party B does not want to have even one leader, so it prefers to move to $(2,0)$. $(2,0)$ is not optimal for party A and it moves to $(1,0)$ (step 2), so we get into a cycle fully characterized by

$$\frac{1+s}{6} \geq C > \max\{\frac{4r+s}{6}, \frac{3sr}{6(r+t)} + \frac{4tr+ts}{6(r+s)}, \frac{1-r}{3} - \frac{st}{3(r+t)} + \frac{tr}{3(r+s)}\}$$

.

7. Once we arrive here, we now from step 3 that $MB_A(2,1) \geq C$ and since $MB_A(2,2) \geq MB_A(2,1)$, $MB_A(2,2) \geq C$ too. Then, no one wants to reduce its number of leaders. If $(2,2)$ is an equilibrium, we are done. Otherwise, two things may happen, either $C < \frac{1}{6}$ and $(3,3)$ is an equilibrium or $\frac{rs}{2(r+t)} + \frac{rt}{2(r+s)} \geq C > \frac{1}{6}$, so that party B wants to deviate but we do not reach an equilibrium (even if through this way we have not analyzed the avalaibility of $(1,2)$ this is not possible because $MB_A(2,2) \geq C$. This situation of lack of equilibria is characterized by:

$$\min\{\frac{1+s}{6}, \frac{rs}{2(r+t)} + \frac{rt}{2(r+s)}\} \geq C > \frac{1}{6}$$

8. $(1,2)$ may be an equilibrium. If not since $MB_A(2,2) \geq MB_B(1,3)$, in case of adding more leaders they will always go to party A and we will go to step 9. If $MB_A(1,2) < C$, the consequent deviations of parties would let them to $(1,1)$. We would be in a cycle and we would not have equilibrium. Through this way we have not analyze the possibility of $(2,1)$ or $(3,3)$ being equilibria, but since $MB_A(1,2) \geq MB_A(2,1)$, $MB_A(1,2) \geq MB_A(3,3)$ and through this way $MB_A < C$, that would be not possible. This lack of equilibria is characterized by:

$$\min\{\frac{1+4r+2s}{6}, \frac{2-r}{6} - \frac{sr}{3(r+t)} + \frac{2tr}{6(r+s)}\} \geq C > \frac{r}{6} + \frac{sr+3st}{6(r+t)} + \frac{tr+3ts}{6(r+s)}$$

9. If $(2,2)$ is an equilibrium we are done. If $MB_A(3,2) \geq C$ and, so, going to $(3,2)$ is a good deviation for A, then $(3,3)$ is an equilibrium. If not the only case is that $MB_B(2,2) < C$ and party B moves to $(2,1)$. $MB_A(2,1)$ or $MB_B(2,1)$ are smaller than C, $(2,1)$ would not be an equilibrium either and the equilibrium would not exist. This case is characterized by:

$$\min\{\frac{1+4r+2s}{6}, \frac{1-r}{3} - \frac{st}{3(r+t)} + \frac{tr}{3(r+s)}\} \geq C > \frac{1}{6}$$

and

$$\min\{\frac{1+s}{6}, \frac{4r+s}{6}\} < C$$

The first clear conclusion that could be derived is that there are many more cases compared to the centrist biased population, where the equilibria does not exist. Furthermore, even if it existed we could not ensure that such equilibrium would be unique. Therefore, it is clear that these are not two sides of the same coin. The existance of equilibrium becomes a harder task when one of the populations is biased to the extreme rather than when it is biased to the center.

However, the possible equilibria are not as much as in the previous subsection. Without taking into account the equilibria where the two parties have the same number of leaders, there are only three other possible equilibria: two where party A has more leaders, $(1,0)$ and $(2,1)$ and one where party B has more leaders $(2,1)$.

3.3 Examples

We have constructed some examples with specific distributions of $\{r,s,t\}$ in order to see which are the possible equilibria, if any, for different values of C. We give two different distributions for each of the cases: the uniform population against the centrist biased one and the uniform population against the extremist biased one. The exact distributions of the centrist and the extremist biased populations are symmetric, so that we can also make comparisons between the two different frameworks.

Consider as a first case $\{r, s, t\} = \{\frac{4}{7}, \frac{2}{7}, \frac{1}{7}\}$. Then, we would have these possible equilibria: $(0, 0)$ for $C > \frac{1}{2}$; $(0, 1)$ for $C \in (\frac{5}{14}, \frac{1}{2}]$; $(1, 1)$ for $C \in (\frac{3}{14}, \frac{5}{14}]$; $(2, 1)$ for $C \in (\frac{13}{70}, \frac{3}{14}]$; no equilibria for $C \in (\frac{58}{315}, \frac{13}{70}]$; $(2, 2)$ for $C \in (\frac{1}{6}, \frac{58}{315}]$; $(3, 2)$ for $C \in (\frac{11}{210}, \frac{1}{6}]$ and $(3, 3)$ for $C \leq \frac{11}{210}$ (see figure 4).

Consider, as a second example a not so centrist biased population as the first one, $\{r, s, t\} = \{\frac{7}{20}, \frac{33}{100}, \frac{8}{25}\}$. The possible equilibria would be the following: $(0, 0)$ for $C > \frac{1}{2}$; $(0, 1)$ for $C \in (\frac{49}{100}, \frac{1}{2}]$; $(1, 1)$ for $C \in (\frac{74843}{341700}, \frac{49}{100}]$; $(1, 2)$ for $C \in (\frac{7477}{34170}, \frac{74843}{341700}]$; $(2, 2)$ for $C \in (\frac{1}{6}, \frac{7477}{34170}]$; $(3, 2)$ for $C \in (\frac{891}{5695}, \frac{1}{6}]$; $(3, 3)$ for $C \leq \frac{891}{5695}$ (see figure 5).

The main difference between these two examples is that the latter is not so biased to the extreme. Therefore there are more values of C for which the equilibrium is a pair with the same number of leaders for both parties. For some large values of C the centrist biased party has more leaders, in $(0, 1)$ and $(1, 2)$, but for small values of C, because of the greater efficience of the centrist leaders. We can also see the contrary effect when C becomes smaller so that the first party obtaining a third leader is the one with the uniform population. In the first example, the range of values of C for which both parties have the same number of leaders is not so large, specially for $(2, 2)$ and $(3, 3)$. We can still observe that the centrist biased party is the first one getting a leader, but it is not the case when obtaining the second or the third leader. Notice that we may also find values of C for which the equilibrium does not exist.

A third example would be a similar distribution to the first one, but applied to the case of the uniform population against the extremist biased one, that is $\{r, s, t\} = \{\frac{1}{7}, \frac{2}{7}, \frac{4}{7}\}$. In this case the possible equilibria would be: $(0, 0)$ for $C > \frac{1}{2}$; $(1, 0)$ for $C \in (\frac{5}{14}, \frac{1}{2}]$; $(1, 1)$ for $C \in (\frac{86}{315}, \frac{5}{14}]$; $(1, 2)$ for $C \in (\frac{169}{630}, \frac{86}{315}]$; no equilibria for $C \in (\frac{1}{6}, \frac{169}{630}]$ and $(3, 3)$ for $C \leq \frac{1}{6}$ (see figure 6).

And the fourth exameple, related to the second one but, again, in the case of the extremist biased population, that is $\{r, s, t\} = \{\frac{8}{25}, \frac{33}{100}, \frac{7}{20}\}$. We have the following possilbe equilibria: $(0, 0)$ for $C > \frac{1}{2}$,$(1, 0)$ for $C \in (\frac{49}{100}, \frac{1}{2}]$; $(1, 1)$ for $C \in (\frac{59221}{261300}, \frac{49}{100}]$; $(1, 2)$ for $C \in (\frac{14699}{65325}, \frac{59221}{261300}]$; no equilibria for $C \in (\frac{116693}{522600}, \frac{14699}{65325}]$; $(2, 2)$ for $C \in (\frac{7623}{43550}, \frac{116693}{522600}]$; no equilibria again for $C \in (\frac{1}{6}, \frac{7623}{43550}]$ and $(3, 3)$ for $C \leq \frac{1}{6}$ (see figure 7).

We can see in the third and the fourth examples that these are not just copies of the first and the second. However we may find some of the phenomena here as well. Obviously, the population which is more biased to the center, or less biased to the extreme, the uniform one in this case, will be the one obtaining the first leader. The values of C for which an equilibrium with the same number of leaders for both parties exist is larger for the no so biased distribution. Althought we find two separate segments where the equilibrium does not exist in the not so biased distribution, the set of C for which the equilibrium exists is larger in this framework than in the more biased one.

We can also derive from comparing the first and second examples with the third and fourth respectively, that the equilibrium exists for a larger set of C when we have a uniform population against a centrist biased one, than when it is facing an extremist biased one. Remember that even if it is not reflected in these examples, multiplicity of equilibria may also arise when parties have an internal structure and one of them is biased to the extreme.

4 Conclusion

In this paper, I have constructed a model that endogenously explains how two parties in competition choose their respective optimal number of leaders. It is assumed that parties have separate ideology platforms that do not overlap. Thus, they do not compete for the same voters. Instead of that, they try to optimally reduce their respective abstention. Separating the population in two groups allows us to examine which are the factors that explain why one of the parties might have more leaders than the other one.

First of all, I have fully characterized the equilibrium for the symmetric case where the two separable populations are uniformly distributed. In this case there is a unique equilibrium, where both parties receive the support of the same number of leaders, a number which is decreasing in the cost-benefit ratio of leadership. This result matches the one by Herrera and Martinelli [?]. From the comparative statics, we have seen the first factor that may explain a different number of leaders between parties. If one of them has a smaller cost-benefit ratio (due to cheaper communication channels or higher expected benefits from wining the elections), it will have more leaders in equilibrium. Similarly, if the leaders of one party are more efficient when influencing their followers or, equivalently, if their party has more potential voters, there could be equilibria where such party received the support of a greater number of leaders.

Although I have not been able to develop theoretical conclusion about general distributions, I have extracted some conclusions from the wide family of beta distributions. With respect to the first leader for each of the parties, the party that has a distribution of potential voters with a mean closer to the center than the median obtains it first. This is described in the paper for beta distributions. The conclusions extracted for the second leader seem to be valid for any number of leaders, although due to difficulties in computation I have not been able to check for this. In case of symmetric populations for each of the two parties ($\alpha = \beta$), the one with the smaller variance obtain a higher marginal benefit and, therefore, will be ready to have more leaders. In case of non-symmetric beta distributions for each of the two parties, the parameter α (resp. β) increases (resp. *decreases*) the marginal benefit for the first leader, while it decreases (resp. increases) the marginal benefit for the second one (supposedly for each of the following ones).

Finally, the introduction of an internal structure together with the asymmetric distribution of potential voters can also explain one party having more leaders than the other. This is what I have examined in the third section of the paper by using a discrete interpretation of the model. In this approach it is assumed the preexistence of a few groups in the party such that all the individuals in that group coordinate in the same action. In this setup the party that has an structure more biased to the center has an advantage, because they are expected to be, initially, more efficient. Even if discrete treatment opens the door for non existence and multiplicity of equilibria in some cases it also lightens how asymmetric distributions justify different number of leaders among parties through these internal structures. We can find equilibria where the most centrist biased party has more leaders, which hold for large values of the leadership cost. However, we can also find some equiibria where the disadvantaged party (the one with a structure biased to the extreme) has more leaders. The latter kind of equilibria hold for small values of the leadership cost. This effect is very important, because there are cases where the initially disadvantaged party has a greater probability of winning the elections, thanks to its greater number of leaders.

We have found some different factors that may explain why two parties may be supported by a different number of leaders. However, there are still some of them that would need to be considered. Given the influence mechanism that we have used, convincing to a potential leader of one party to vote for the opposite one was not possible. In case it was, the party with the most centrist biased distribution would have an extra benefit that could justify the optimal support of a greater number of leaders. Some would also like to include the assumption that the ideological extremists would be more ready to become leaders as the consequence of their greater fear of the opposition's government. This would induce an advantage to the parties with a distribution biased to the extreme.

Some people may it find interesting allowing the parties not only to choose the number of leaders, but also their ideological location. It is straightforward that without decreasing influence functions, we would find an equilibrium with a single leader from each party located at the mediam. Decreasing influence functions would only imply (in almost all cases) more leaders who are ideologically equally distanced from each other.

For future research this could be applied to distributions of population over the subjective perception of the parties platform. With simulations of repetitions of appearances. of leaders, the probability of the stochastic order can easily be approximated. Thus, we have a proxy of the marginal benefits for any countable number of leaders in an environment of two parties. With the continuous approach the unique simmetric equilibria could be predicted, while if we accepted internal structures the prediction is less clear. Some mixture of the two systems, depending if citizens accept to follow the advices of some specific ideological movement (such there is total coordination inside the group) could also be interesting. Since the leadership costs are hard to compute, what we could predict is the evolution of it. If we are in a current situation with two leaders in the leftist party and three in the rightist, and if the cost benefit ratio of leadership raised, we would predict the next stituation.

A welfare analysis could also be interesting. We could do that from the point of view of citizens, by specifying how is going to be constructed the final policy, how does it change with the different leadership costs, distribution or internal structures, and ordering them with respect to the preferences of the median voeter. We could also analyze which are parties' preferences over some common leadership cost and/or if they are ready to modify their internal structure.

References

[1] Besley, T and S. Coate. (1997) An Economic Model of Representative Democracy. Quarterly Journal of Economics. Vol 108. Issue 1. pp. 85-114.

[2] Brooks, C; J. Manza and P. Nieuwbeerta. (2006) Cleavage-based voting behavior in cross national perspective: evidence from six postwar democracies. Social Science Research. Vol 35. Issue 2, pp 88-138.

[3] Gelman, A; G. King and W. J. Boscarding. Estimating the Probability of Events That have Never Ocurred: When is Your Vote Decisive?

[4] Feddersen, T. J. (2004) Rational Choice Theory and the Paradox of Not Voting. Journal of Economic Perspectives. Vol 18. No 1, pp 99-112.

[5] Glaeser, E; B. Sacerdote and J. A. Scheinkman. (1996) Crime and Social Interactions. The Quaterly Journal of Economics. Vol 11. Issue 2. pp. 507-548.

[6] Lazarsfeld, P.; B. Berelson and H. Gaudet. (1948) The People's Choice. Columbia University Press. New York.

[7] Lazarsfeld, P. F. and R.K. Merton. (1954) Friendship as a Social Process: A Substantive and Methodological Analysis. In Freedom and Control in Modern Society, Morroe Berger, Theodore Abel, and Charles H. Page, eds. New York.

[8] Martinelli, C. and H. Herrera. (2005) Group Formation and Voter Participation. Theoretical Economics 1, pp 461-487.

[9] Mattozzi, A. and A. Galeotti. (2007) Personal Influence: Social Context and Political Competition.

[10] Precker, J. A. (1952) Similarity of Valuings as a Factor in Selection of Peers and Near-Authority Figures. Journal of Abnormal and Social Psichology. Vol.47, pp. 406-414.

[11] Quattrone, G. and A. Tverky. (1988) Contrasting Rational and Psychological Analyses of Political Choice. The American Political Science Review. Vol.82, No. 3, pp. 719-736.

[12] Riker, W. and P. Ordeshook. (1968) A Theory of the Calculus of Voting. American Political Science Review. March 62, pp. 25-42.

[13] Robinson, J. P. (1976) Interpersonal Influence in Election Campaigns: Two Setp-flow Hypotheses. The Public Opinion Quaterly, Vol. 40. No. 3.

[14] Roper, E. (1955) Foreword to Personal Influence by E. Katz and P. F. Lazarsfeld. Transaction Publishers, New Brunswick, New Jersey.

[15] Shachar, R. and B. Nalebu?. (1999) Follow the Leader: Theory and Evidence on Political Participation. American Economic Review Vol. 89, No. 3, pp. 525-547 .

FIGURES

Figure 1: Areas of influence of each lider

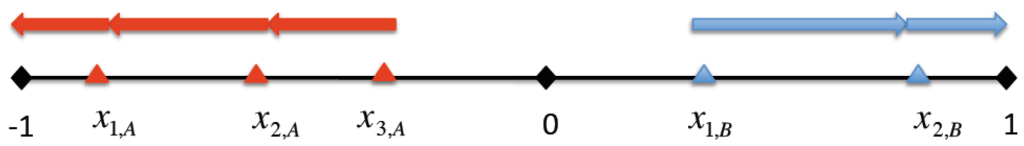


Figure 2: Vote share of the first leader for $Beta(\alpha, \beta)$

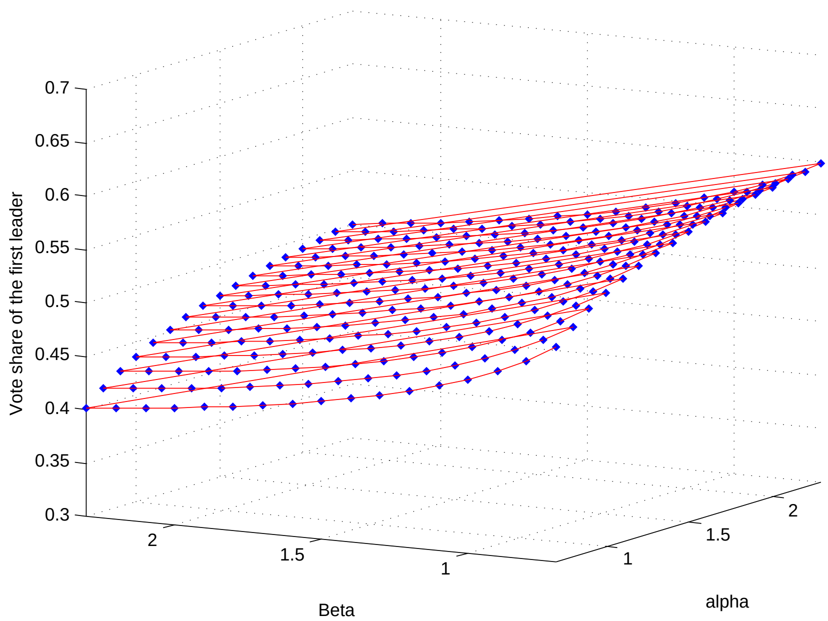


Figure 3: Vote share of the second leader for $Beta(\alpha, \beta)$

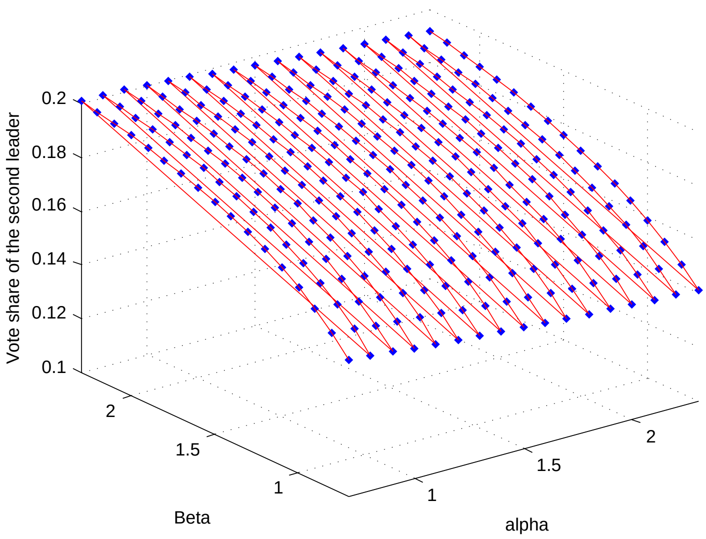


Figure 4: Example 1: $\{r, s, t\} = \{\frac{4}{7}, \frac{2}{7}, \frac{1}{7}\}$

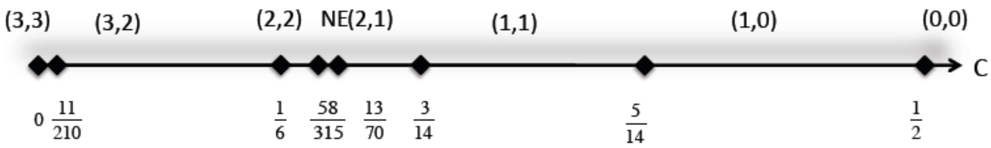


Figure 5: Example 2: $\{r, s, t\} = \{\frac{7}{20}, \frac{33}{100}, \frac{8}{25}\}$

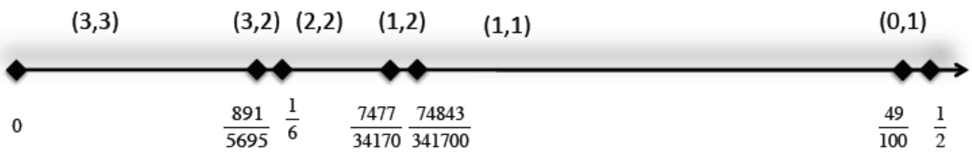


Figure 6: Example 3: $\{r, s, t\} = \{\frac{1}{7}, \frac{2}{7}, \frac{4}{7}\}$

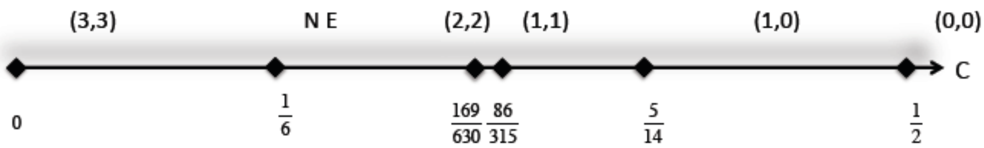
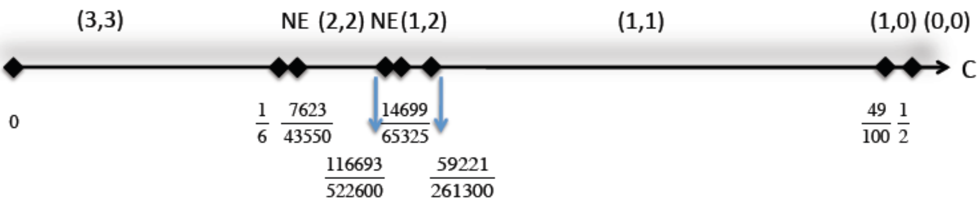


Figure 7: Example 4: $\{r, s, t\} = \{\frac{8}{25}, \frac{33}{100}, \frac{7}{20}\}$



APPENDIX

Proof of Lemma 1

Take without loss of generality the leftist party.

$$\frac{MB_A(L_A+1,L_B)}{MB_A(L_A,L_B)} = \frac{L_B/[(L_A+L_B+1)(L_A+L_B)]}{L_B/[(L_A+L_B)(L_A+L_B-1)]}$$
$$= \frac{L_A+L_B-1}{L_A+L_B+1} < 1$$

Proof of Lemma 2

Take without loss of generality the leftist party.

$$\frac{MB_A(L_A,L_B+1)}{MB_A(L_A,L_B)} = \frac{(L_B+1)/[(L_A+L_B+1)(L_A+L_B)]}{L_B/[(L_A+L_B)(L_A+L_B-1)]} = \frac{(L_B+1)(L_A+L_B-1)}{L_B(L_A+L_B+1)}$$

If $\frac{MB_A(L_A,L_B+1)}{MB_A(L_A,L_B)} \geq 1$, it must be the case that:

$$(L_B+1)(L_A+L_B-1) \geq L_B(L_A+L_B+1)$$
$$L_BL_A + L_B^2 - L_B + L_A - L_B - 1 \geq L_AL_B + L_B^2 - L_B$$
$$L_A \geq L_B + 1$$

Proof of Lemma 3

First we can analyze $(1,0)$ (and equivalently $(0,1)$). Party A would never increase its number of leaders, once in $(1,0)$, since it is already winning the elections for sure. As we have seen in the requirement for equilibrium it must be the case that $MB_A(1,0) \geq C$. We have seen that $MB_A(1,0) = \frac{1}{2}$, so it needs to be the case that $C \leq \frac{1}{2}$. However, we should also check for the optimality of party B. At $(1,0)$ party B is losing the elections for sure, while at $(1,1)$ its probability of winning is $\frac{1}{2}$. Then it is easy to conclude that $MB_B(1,1) = \frac{1}{2}$. In order $(1,0)$ to be an equilibrium such move cannot be optimal for party B, so it must be the case that $C > \frac{1}{2}$. This is a contradiction with the previous requirement, so it cannot be the case.

Clearly, no equilibrium of the form $(L,0)$ or $(0,L)$ can exist for $L > 1$, because the party with a positive number of leaders would always be ready to decrease, at least by one, the number of leaders. That would imply

a smaller cost, without disminishing the probability of winning, which is equal to 1. Hence, any situation where there are a strictly positive number of leaders for one party and no leaders for the other one, cannot constitute an equilibrium.
The first part of the proof has already given before. That is, no equilibrium of the form $(L, 0)$ or $(0, L)$ exists for any $L \geq 0$. Now, we will prove that no equilibrium exists when both parties have a positive number of leaders and one has more leaders than the other. In order to do that, take without loss of generality a pair of leaders $(L+j, L)$, where j is a strictly positive integer and check for the equilibrium conditions.

In order to be optimal for party A, it must be the case that $MB_A(L+j, L) \geq C$ and $MB_A(L+j+1, L) < C$. That means:

$$\frac{L}{(2L+j)(2L+j-1)} \geq C \text{ and } \frac{L}{(2L+j+1)(2L+j)} < C$$

And in order to be optimal for party B, we would equivalently ask that $MB_B(L+j, L) \geq C$ and $MB_B(L+j, L+1) < C$, which is resumed in:

$$\frac{L+j}{(2L+j)(2L+j-1)} \geq C \text{ and } \frac{L+j}{(2L+j+1)(2L+j)} < C$$

However, $MB_B(L+j, L+1) \geq MB_A(L+j, L)$, as it is proved here:

$$\frac{L+j}{(2L+j+1)(2L+j)} - \frac{L}{(2L+j)(2L+j-1)} \geq 0$$

$$\frac{(L+j)(2L+j-1) - L(2L+j+1)}{(2L+j+1)(2L+j-1)} \geq 0$$

Since the two terms in the denominator are always positive the inequality holds as long as the numerator is positive. So we ask that:

$$(L+j)(2L+j-1) - L(2L+j+1) = 2L^2 + jL - L + 2Lj + j^2 - j - 2L^2 - jL - L \geq 0$$

$$2Lj + j^2 - j - 2L \geq 0$$

$$2L(j-1) + j(j-1) \geq 0$$

which holds for $j \geq 1$, so $MB_B(L+j+1, L) \geq MB_A(L+j, L) \geq C$. Then it cannot be true that $MB_B(L+j+1, L) < C$.

Proof of Proposition 2

Assume, without loss of generality that $C_A > C_B$ and that (L_A, L_B) is an equilibrium. Then, the marginal benefit of party A has to be greater than C_A, that is: $MB_A(L_A, L_B) = \frac{L_B}{(L_A+L_B)(L_A+L_B-1)} \geq C_A$. At the same time, the marginal benefit of party B has to be greater than C_B; but, not only that, the marginal benefit, having one more leader has to be strictly smaller than C_B, that is: $Q_B(L_A, L_B+1) = \frac{L_A}{(L_A+L_B+1)(L_A+L_B)} < C_B$.

If $C_A > C_B$ it has to be the case that $MB_A(L_A, L_B) > MB_B(L_A, L_B + 1)$. That is:

$$\frac{L_B}{(L_A + L_B)(L_A + L_B - 1)} > \frac{L_A}{(L_A + L_B)(L_A + L_B + 1)}$$

The preceeding inequality only holds if $L_B^2 - L_B - L_A^2 + L_A > 0$, or, equivalently, if $L_B^2 - L_A^2 > L_B - L_A$. This condition can also be expressed this way:

$$(L_B + L_A)(L_B - L_A) > L_B - L_A$$

This inequality cannot hold for $L_A > L_B$. Both sides of the inequality would be negative and the LHS would be obviously greater in absolute value. Therefore, in equilibrum it needs to be the case that $L_B \geq L_A$.

Soving the integral

$$\int_0^\alpha \int_{1-\frac{x_{1,B}}{\alpha}}^1 \frac{L_A L_B z_{1,A}^{L_A-1}}{\alpha}(1 - \frac{x_{1,B}}{\alpha})^{L_B-1} dz_{1,A} dx_{1,B} =$$

$$\int_0^\alpha (1 - \frac{x_{1,B}}{\alpha})^{L_B-1} \frac{L_B}{\alpha}[1 - (1 - \frac{x_{1,B}}{\alpha})^{L_A}] dx_{1,B} =$$

$$\int_0^\alpha (1 - \frac{x_{1,B}}{\alpha})^{L_B-1} \frac{L_B}{\alpha} dx_{1,B} - \int_0^\alpha (1 - \frac{x_{1,B}}{\alpha})^{L_A+L_B-1} \frac{L_B}{\alpha} dx_{1,B} =$$

$$-\frac{(1 - \frac{0}{\alpha})^{L_B}}{L_B \frac{-1}{\alpha}} \frac{L_B}{\alpha} + \frac{(1 - \frac{0}{\alpha})^{L_A+L_B}}{(L_A + L_B)\frac{-1}{\alpha}} \frac{L_B}{\alpha} =$$

$$1 - \frac{L_B}{L_A + L_B} = \frac{L_A}{L_A + L_B}$$

Proof of Proposition 2

Assume that the A party has L leaders and the B party has $L+j$ leaders. If that is an equilibrium it must be the case that $MB_B(L, L+j) \geq C$ and, among others, that $MB_A(L+1, L+j) < C$. We will compute:

$$MB_A(L{+}1, L{+}j) - MB_B(L, L{+}j) = \frac{(L+j)[1+(\beta-1)(L+1)+(\beta-1)(L+j)]}{(2L+j+1)\beta} - \frac{L}{2L+j-1}$$

$$= \frac{(L+j)(2L+j-1)[1+(\beta-1)(L+1)+(\beta-1)(L+j)] - L(2L+j+1)\beta}{(2L+j+1)(2L+j-1)\beta}$$

Which is weakly greater than 0 as long as the numerator is so. Then we ask that:

$$(L+j)(2L+j-1)[1+(\beta-1)(L+1)+(\beta-1)(L+j)] - L(2L+j+1)\beta \geq 0$$

$$(2L^2+jL-L+2Lj+j^2-j)[\beta+2L(\beta-1)+j(\beta-1)] - \beta[2L^2+(j+1L)] \geq 0$$

$$[2L^2+L(3j-1)+j(j-1)][\beta+2L(\beta-1)+j(\beta-1)] - \beta[2L^2+(j+1L)] \geq 0$$

Which is true because for $j \geq 1$ and $j(j-1) \geq 0$.

Calculus of $E[|x_{i+1,j} - x_{i,j}|]$, where $x_{i,j}, x_{i+1,j} \sim U(0, \alpha)$

$$E[|x_{i+1,j} - x_{i,j}|] = \int_0^{\alpha}\int_0^{x_{i+1,j}} (x_{i+1,j} - x_{i,j}) \frac{L_j!(\frac{x_{i,j}}{\alpha})^{i-1}(1-\frac{x_{i+1,j}}{\alpha})^{L_j-i-1}}{\alpha^2(i-1)!(L_j-i-1)!}$$

$$= \int_0^{\alpha}\int_0^{x_{i+1,j}} \frac{L_j! x_{i+1,j}^{i+1}(\alpha - x_{i+1,j})^{L_j-i-1}}{\alpha^{L_j} i!(L_j-i-1)!} - \int_0^{\alpha}\int_0^{x_{i+1,j}} \frac{L_j! x_{i+1,j}^{i+1} i(\alpha - x_{i+1,j})^{L_j-i-1}}{\alpha^{L_j}(i+1)!(L_j-i-1)!}$$

$$= \int_0^{\alpha} \frac{x_{i+1,j}^{L_j}(i+1)}{\alpha^{L_j}} - \int_0^{\alpha} \frac{i x_{i+1,j}}{\alpha^{L_j}} = \frac{\alpha(i+1)}{L_j-1} - \frac{\alpha i}{L_j-1} = \frac{\alpha}{n+1}$$

$(1,1)$ and $(2,2)$ can be equlibrium when $r > s > t$

$(1,1)$ can be an equilibrium, if $MB_A(1,1) = \frac{1+2s+4t}{6} \geq C$, $MB_B(1,1) = \frac{1+4r+2s}{6} \geq C$, $MB_A(2,1) = \frac{1+s}{6} < C$ and $MB_B(1,2) = \frac{1}{3} - \frac{r}{3} - \frac{2st}{6(r+t)} + \frac{2tr}{6(r+s)} < C$. We can easily verify this assertion by assuming for instance $t = 0$ and $s = \frac{1}{4}$. Since, $\min\{MB_A(1,1), MB_B(1,1)\} = \min\{\frac{1}{4}, \frac{3}{4}\} > \max\{MB_A(2,1), MB_B(1,2\} = \max\{\frac{5}{24}, \frac{1}{12}\}$, there are values of C for which the equilibrium exists.

$(2,2)$ can be an equilibrium as well, if $MB_A(2,2) = \frac{2-r}{6} - \frac{sr}{6(r+t)} - \frac{tr}{6(r+s)} \geq C$, $MB_B(2,2) = \frac{3sr}{6(r+t)} + \frac{4tr+ts}{6(r+s)} \geq C$, $MB_A(3,2) = \frac{1}{6} < C$ and

$MB_B(2,3) = \frac{4(1-r)}{6} - \frac{4sr+st}{6(r+t)} - \frac{4tr+ts}{6(r+s)} < C$. One case where this may be true is when $\{r,s,t\} = \{\frac{1}{2}, \frac{1}{3}, \frac{1}{6}\}$. Since, $\min\{MB_A(2,2), MB_B(2,2)\} = \min\{\frac{23}{120}, \frac{73}{360}\} > \max\{MB_A(3,2), MB_B(2,3\} = \max\{\frac{1}{6}, \frac{3}{40}\}$, there are values of C for which the equilibrium exists.

$(2,1)$**,** $(3,1)$ **and** $(3,2)$ **can be equlibrium when** $r > s > t$

$(2,1)$ can be an equilibrium if $MB_A(2,1) = \frac{1+s}{6} \geq C$, $MB_B(2,1) = \frac{4r+s}{6} \geq C$, $MB_A(3,1) = \frac{r+s}{6} < C$ and $MB_B(2,2) = \frac{3sr}{6(r+t)} + \frac{4tr+ts}{6(r+s)} < C$. Take, for instance $\{r,s,t\} = \{\frac{9}{10}, \frac{1}{10}, 0\}$. Since, $\min\{MB_A(2,1), MB_B(2,1)\} = \min\{\frac{11}{60}, \frac{37}{60}\} > \max\{MB_A(3,1), MB_B(2,2\} = \max\{\frac{1}{6}, \frac{1}{20}\}$, there are values of C for which the equilibrium exists.

$(3,1)$ can be an equilibrium if $MB_A(3,1) = \frac{r+s}{6} \geq C$, $MB_B(3,1) = \frac{r}{2} \geq C$ and $MB_B(3,2) = \frac{rs}{2(r+t)} + \frac{rt}{2(r+s)} < C$. One example would be defining $\{r,s,t\} = \{\frac{3}{4}, \frac{1}{4}, 0\}$. $\min\{MB_A(3,1), MB_B(3,1)\} = \min\{\frac{1}{6}, \frac{3}{8}\} < MB_B(3,2) = \frac{1}{8}$.

$(3,2)$ can be an equilibrium as well. For that, it must be that $MB_A(3,2) = \frac{1}{6} \geq C$, $MB_B(3,2) = \frac{rs}{2(r+t)} + \frac{rt}{2(r+s)} \geq C$ and $MB_B(3,3) = \frac{st}{2(r+t)} + \frac{st}{2(r+s)}$. If we set $t = 0$, $MB_B(3,3) = 0$, while the marginal probabilities for both parties at $(3,2)$ are strictly positive, so there exist values of C for which $(3,2)$ is an equilibrium.

$(1,2)$ **can be an equilibrium when** $r > s > t$

In order to be so, we must have $MB_A(1,2) = \frac{r}{6} + \frac{sr+3st}{6(r+t)} + \frac{tr+3ts}{6(r+s)} \geq C$, $MB_B(1,2) = \frac{1-r}{3} - \frac{2st}{6(r+t)} + \frac{2tr}{6(r+s)} \geq (1,2)C$, $MB_A(2,2) = \frac{2-r}{6} - \frac{sr}{6(r+t)} - \frac{tr}{6(r+s)} < C$ and $MB_B(1,3) = \frac{1-r}{2} - \frac{3sr+st}{6(r+t)} - \frac{3tr+ts}{6(r+s)} < C$. Take, for instance $\{r,s,t\} = \{\frac{7}{20}, \frac{33}{100}, \frac{8}{25}\}$. Since $\min\{MB_A(1,2), MB_B(1,2)\} > \max\{MB_A(2,2), MB_B(1,3)\}$, there are values of C for which $(1,2)$ holds as equilibrium.

Equilibria that cannot coexist when $r > s > t$

$(0,1)$ and $(2,2)$ cannot be equilibrium at the same time. Two of the conditions that need to be fulfilled are $MB_A(1,1) < C$ and $MB_B(2,2) \geq C$. These two cannot hold at the same time because $MB_A(1,1) > MB_B(2,2)$. This is easy to check if we rewrite $MB_A(1,1) = \frac{1+2s+4t}{6} = \frac{r+3s+5t}{6}$. Then, it is obvious that:

$$MB_A(1,1) = \frac{r+3s+5t}{6} = \frac{r}{6} + \frac{3sr+3st}{6(r+t)} + \frac{5tr+5ts}{6(r+s)} > \frac{3sr}{6(r+t)} + \frac{4tr+ts}{6(r+s)} = MB_B(2,2)$$

$(0,1)$ and $(2,1)$ cannot coexist as equilibrium. If they did, two of the conditions that would need to hold are $MB_A(1,1) < C$ and $MB_A(2,1) \geq C$. However, this is not possible because, clearly, $MB_A(1,1) = \frac{1+2s+4t}{6} \geq \frac{1+s}{6} = MB_A(2,1)$.

$(2,1)$ and $(1,2)$ cannot be equilibrium at the same time. Two of the conditions that would need to be fulfilled are $MB_B(2,2) < C$ and $MB_B(1,2) \geq C$. These two inequations cannot be true at the same time because $MB_B(2,2) \geq MB_B(1,2)$, that is: $\frac{3sr}{6(r+t)} + \frac{4tr+ts}{6(r+s)} \geq \frac{1}{3} - \frac{r}{3} - \frac{2st}{6(r+t)} + \frac{2tr}{6(r+s)}$. If we reorganize and simplify this inequality we will get to:

$$2r + \frac{3sr + 2st}{r+t} + \frac{2tr + ts}{r+s} \geq 2$$

Take the left hand side of this inequation and check that:

$$2r + \frac{3sr + 2st}{r+t} + \frac{2tr + ts}{r+s} \geq 2r + \frac{2sr + 2st}{r+t} + \frac{sr}{r+s} + \frac{2tr + ts}{r+s} \geq 2r + \frac{2sr + 2st}{r+t} + \frac{2tr + 2ts}{r+s}$$

Since the right hand side of the last inequation is equal to 2, it is proved that $MB_B(2,2) \geq MB_B(2,1)$.

$(2,1)$ and $(3,3)$ cannot be equilibrium at the same time either. This is because if they were so, $MB_B(2,2) < C$ and $MB_B(3,3) \geq$ would need to hold. But these two inequalities cannot hold simoultaneously because $MB_B(2,2) = \frac{3sr}{6(r+t)} + \frac{4tr+ts}{6(r+s)} \geq \frac{3st}{6(r+t)} + \frac{3st}{6(r+s)} = MB_B(3,3)$. This inequality comes straightforward from $r > s > t$.

The last two equilibrias that need to be compared and that cannot coexist are $(3,1)$ and $(3,3)$. Otherwise, $MB_B(3,2) < C$ and $MB_B(3,3) \geq C$ would need to be true at the same time, but this is impossible because, obviously $MB_B(3,2) = \frac{rs}{2(r+t)} + \frac{rt}{2(r+s)} \geq \frac{st}{2(r+t)} + \frac{st}{2(r+s)}$.

$(1,1)$ **and** $(2,2)$ **can be equilibrium when** $r < s < t$

$(1,1)$ can be an equilibrium if $MB_A(1,1) = \frac{1+2s+4t}{6} \geq C$, $MB_B(1,1) = \frac{1+4r+2s}{6} \geq C$, $MB_A(2,1) = \frac{1+s}{6} < C$ and $MB_B(1,2) = \frac{1}{3} - \frac{r}{3} - \frac{2st}{6(r+t)} + \frac{2tr}{6(r+s)} < C$. We can easily verify this assertion by assuming for instance $\{r,s,t\} = \{\frac{1}{5}, \frac{2}{5}, \frac{3}{5}\}$. Since, $\min\{MB_A(1,1), MB_B(1,1)\} = \min\{\frac{17}{30}, \frac{13}{30}\} > \max\{MB_A(2,1), MB_B(1,2\} = \max\{\frac{7}{30}, \frac{2}{9}\}$, there are values of C for which the equilibrium exists.

$(2,2)$ can also be an equilibrium if $MB_A(2,2) = \frac{2-r}{6} - \frac{sr}{6(r+t)} - \frac{tr}{6(r+s)} \geq C$, $MB_B(2,2) = \frac{3sr}{6(r+t)} + \frac{4tr+ts}{6(r+s)} \geq C$, $MB_A(3,2) = \frac{1}{6} < C$ and $MB_B(2,3) =$

$\frac{4(1-r)}{6} - \frac{4sr+st}{6(r+t)} - \frac{4tr+ts}{6(r+s)} < C$. We can easily verify this assertion by assuming for instance $\{r, s, t\} = \{\frac{1}{5}, \frac{2}{5}, \frac{3}{5}\}$. Since, $\min\{MB_A(2,2), MB_B(2,2)\} = \min\{\frac{23}{90}, \frac{1}{5}\} > \max\{MB_A(3,2), MB_B(2,3\} = \max\{\frac{1}{6}, \frac{2}{15}\}$, there are values of C for which the equilibrium exists.

Proof of $MB_B(1,2) \geq MB_B(2,2)$

$$MB_B(1,2) = \frac{2}{6} - \frac{2r}{6} - \frac{2st}{6(r+t)} + \frac{2tr}{6(r+s)} \geq \frac{3sr}{6(r+t)} + \frac{4tr+ts}{6(r+s)} = MB_B(2,2)$$

$$2 \geq 2r + \frac{3sr+2st}{r+t} + \frac{2tr+ts}{r+s}$$

$$2s + 2t \geq \frac{3sr+2st}{r+t} + \frac{2tr+ts}{r+s}$$

$$0 \geq \frac{sr}{r+t} - \frac{ts}{r+s} = s(\frac{r}{r+t} - \frac{t}{r+s})$$

which is true because $t > r$ and $r + s < r + t$. Hence, there is no pair of equilibria with the same number of leaders that can coexist.

$(2,1)$ **can be an equilibrium when** $r < s < t$

In order to have $(2,1)$ as an equilibrium it must be the case that $MB_A(2,1) \geq C$, $MB_B(2,1) \geq C$, $MB_A(3,1) < C$ and $MB_B(2,2) < C$. Take for instance $\{r, s, t\} = \{\frac{8}{25}, \frac{33}{100}, \frac{7}{20}\}$. That would make $\min\{MB_A(2,1), MB_B(2,1)\} = \min\{\frac{133}{600}, \frac{161}{600}\} > \max\{MB_A(3,1), MB_B(2,2\} = \max\{\frac{13}{120}, \frac{116693}{522600}\}$. Then, there exist values of C that make $(2,1)$ an equilibrium.

Equilibria that cannot coexist when $r < s < t$

$(1,0)$ and $(2,2)$ cannot be equilibrium at the same time. If they could, $MB_B(1,1) < C$ and $MB_A(2,2) \geq C$. But these two cannot be true at the same time because $MB_B(1,1) \geq MB_A(2,2)$. That is:

$$\frac{1+4r+2s}{6} \geq \frac{2-r}{6} - \frac{sr}{6(r+t)} - \frac{tr}{6(r+s)}$$

$$4r + 2s + r(\frac{s}{r+t} + \frac{t}{r+s}) \geq 1$$

which is obvious if we remember that we have already proved that $\frac{s}{r+t} + \frac{t}{r+s} \geq 1$.

$(1,0)$ and $(3,3)$ cannot coexist as equilibra either, because otherwise $MB_B(1,1) < C$ and $MB_A(3,3) \geq C$ would need to be true. This is clearly impossible since $MB_B(1,1) = \frac{1+4r+2s}{6} \geq \frac{1}{6} = MB_A(3,3)$.

Another two pair of equilibria that cannot coexist are $(1,2)$ and $(2,1)$. This is because two of the conditions that would need to hold are $MB_A(2,2) = \frac{2-r}{6} - \frac{sr}{6(r+t)} - \frac{tr}{6(r+s)} < C$ and $MB_A(2,1) = \frac{1+s}{6} \geq C$. However, $MB_A(2,2) \geq MB_A(2,1)$, since:

$$\frac{2-r}{6} - \frac{sr}{6(r+t)} - \frac{tr}{6(r+s)} \geq \frac{1+s}{6}$$

$$1 \geq r + s + \frac{sr}{r+t} + \frac{tr}{r+s}$$

$$t = \frac{ts+tr}{r+s} \geq \frac{sr}{r+s} + \frac{tr}{r+s} \geq \frac{sr}{r+t} + \frac{tr}{r+s}$$

$(1,2)$ and $(3,3)$ cannot be equilibria at the same time. The reason of this is that two of the conditions that would be needed are $MB_A(3,3) = \frac{1}{6} \geq C$ and $MB_A(2,2) = \frac{2-r}{6} - \frac{sr}{6(r+t)} - \frac{tr}{6(r+s)}$. But it is clear that $MB_A(2,2) \geq MB_A(3,3)$, because:

$$2 - r - \frac{sr}{r+t} - \frac{tr}{r+s} \geq 1$$

$$1 = r + \frac{sr+st}{r+t} + \frac{tr+ts}{r+s} \geq r + \frac{sr}{r+t} + \frac{tr}{r+s}$$