Cooperative Production and Efficiency*

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Abstract

We characterize the sharing rule for which a contribution mechanism achieves efficiency in a cooperative production setting when agents are heterogeneous. This rule differs from the one obtained by Sen for the case of identical agents. We also show for a large class of sharing rules that if Nash equilibrium yields efficient allocations, the production function displays constant returns to scale, a case in which cooperation in production is useless.

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1. Introduction

The Cooperative Production problem arises when n agents use a commonly owned technology to transform inputs into outputs. Output is distributed by means of a *sharing rule,* a function yielding consumption of each agent as a function of inputs. In this model, the literature has considered both adverse selection and moral hazard problems. In this paper we focus on adverse selection.[1] In this setting, preferences cannot be used by the planner to achieve the allocations she wants because either preferences are unknown to her or they are not contractible, i.e. nobody can be convicted on the count of falsifying her own preferences. Thus, the planner cannot achieve directly efficient allocations, so she has to construct a mechanism whose non-cooperative outcomes yield the desired allocations. Corchón and Puy (2002) proved that, for any continuous sharing rule, there is a mechanism whose Nash equilibria yield allocations that are Pareto efficient and where agents receive consumptions dictated by the sharing rule.[2] However, the implementing mechanism is complicated so we examine the performance of a natural mechanism in which each agent decides her own input contribution and receives the consumption dictated by the sharing rule. We refer to this mechanism as a contribution mechanism. Holmstrom (1982) and Fabella (1988) showed in two special cases that such a mechanism does not yield efficient allocations as Nash Equilibria.[3] Sen (1966) showed that a particular mix of the egalitarian and the proportional sharing rules achieves efficiency when all agents are identical. However, when agents are not identical and inputs are heterogeneous, Browning (1983) showed that the contribution mechanism described above achieves efficiency only when the production function fulfills

[1] For moral hazard problems see Holmstrom (1982) and Nandeibam (2003).

[2] They also show that in "classical" economies these allocations exist.

[3] Both assume utility functions quasi-linear in consumption. Holmstrom considers sharing rules which depend on aggregate output and Fabella considers the proportional sharing rule. The motivation of the assumption in Holmstrom is moral hazard but his result can be cast in terms of adverse selection too.

a separability property.

In this paper we delve into the kind of sharing rules for which the contribution mechanism achieves efficiency when the domain of admissible preferences is large enough and inputs are homogeneous. We show that the contribution mechanism achieves efficiency only when the production function is a polynomial of, at most, degree $n-1$ (Theorem 1). This is a generalization of a result by Gradstein (1995) in the framework of Cournot oligopoly model. We also characterize the anonymous sharing rule for which the contribution mechanism yields efficient allocations (Theorem 2). We call this rule the *Incremental Sharing Rule.* This rule has two parts. One part awards each agent with her marginal product as it happens with the celebrated Vickrey-Clark-Groves (VCG) mechanisms. The other part is composed by terms that depend on the contributions of other agents. These terms are chosen such that when the production function is a polynomial of, at most, degree n-1 the incremental sharing rule delivers as much consumption as output. Contrarily to VCG mechanisms the incremental sharing rule yields efficient allocations as Nash equilibrium -not as a dominant strategy equilibrium. It works for a large profile of preferences -not only for quasi-linear ones, and requires a polynomial form of the production function not of the utility functions (see for example Liu and Tian [1999] p. 213).

Finally, we replace the anonymity requirement on the sharing rule by one of the following properties: the consumption of any agent either depends on her own labor contribution and the total labor supply, or is zero when the labor supply of this agent is zero. In those cases, efficiency can only be achieved if the production function displays constant returns to scale (Proposition 1). But this is a case where cooperation does not make much sense because it is like if every agent had access to her own technology and would keep the whole output produced using her own technology. Thus, implementation of these sharing rules requires a different mechanism from the one considered here, possibly, a complex one. This implies that Sen´s result is an artifact of his assumption that agents are identical.

This paper is closely related to a paper by Moulin (2007) who independently provided a similar characterization of the incremental sharing rule (the residual* mechanism as Moulin called it) in the context of cost sharing. Formally neither of the results implies another because the two games are not the same. In our case it is a contribution game associated to an output sharing problem and in his case it is a demand game associated to a cost-sharing game. Leroux (2008) has proved that these two games are different.

2. The Model and the Results

We have n agents that supply labor denoted by l_i, $i \in N = \{1, 2, ...n\}$. Let $l \equiv (l_1, l_2, ..., l_n)$, $l_{-i} \equiv (l_1, ..., l_{i-1}, l_{i+1}, ..., l_n)$, $L \equiv \sum_{j=1}^{n} l_j$, $L_{-i} \equiv \sum_{j \neq i} l_j$, $L_{-ik} = \sum_{j \neq i, j \neq k} l_j$, $L_{-ikm} = \sum_{j \neq i, j \neq k, j \neq m} l_j$, and so on. There is a maximum quantity of labor that any agent can supply, $\bar{l}$.

Agents share a technology that is able to generate a consumption good whose production function is written as $X(L)$. It is assumed to be concave, increasing and differentiable in $[0, n\bar{l}]$ with $X(0) = 0$. The production function displays Constant Returns to Scale if $X(L) = aL$, $a > 0$.

Let x_i be the consumption of i and $x \equiv (x_1, x_2, ..., x_n)$. The pair (x, l) is an *allocation.* An allocation (x, l) is feasible if $\sum_{i=1}^{n} x_i = X(L)$ and $0 \leq l_i \leq \bar{l}$, $i \in N$. The set of feasible allocations is denoted by A.

Each agent, say i, has preferences over consumption and labor representable by a concave and differentiable utility function $U_i = U_i(x_i, l_i)$ which is strictly increasing (resp. decreasing) in the first (second) argument.

Efficient allocations are found by

$$\max \sum_{i=1}^{n} \alpha_i U_i(x_i, l_i) \text{ with } (x, l) \in A \tag{2.1}$$

for given $(\alpha_1, \alpha_2, ..., \alpha_n)$ with $\alpha_i \geq 0$ and $\sum_{i=1}^{n} \alpha_i = 1$. This is the maximization of a continuous function over a compact set and, hence, it has a solution by Weierestrass theorem. The program is concave and thus first order conditions gives the maximum. Assuming that first order conditions hold with equality, we have that

$$\frac{\partial U_i(x_i, l_i)}{\partial x_i} \frac{dX(L)}{dL} + \frac{\partial U_i(x_i, l_i)}{\partial l_i} = 0, \ i \in N. \tag{2.2}$$

Let us remark that first order conditions may hold with equality at points where $l_i = 0$. A *Sharing Rule* specifies the consumption allocated to each agent as a function of labor inputs. Formally, a sharing rule, $x(\cdot)$, is a collection of differentiable functions $(x_1(\cdot), x_2(\cdot), ..., x_n(\cdot))$ with $x_i : \Re^n_+ \rightarrow \Re_+$, $i \in N$, such that $\sum_{j=1}^n x_j(l) = X(L)$, $\forall l \in [0, \bar{l}]^n$. Two well-known examples of sharing rules are:

$$x_i^P(l) = \frac{l_i}{\sum_{j=1}^n l_j} X(L), \text{ for all } i \in N \qquad \text{(Proportional)}$$

$$x_i^E(l) = \frac{1}{n} X(L), \text{ for all } i \in N \qquad \text{(Equal Sharing)}$$

Assume that a planner wants to achieve efficient allocations but she cannot observe preferences. The planner can observe the technology and chooses the sharing rule. The planner has to rely on a mechanism whose non-cooperative outcomes yield the desired allocations. The mechanism consists in a message space and an outcome function mapping messages into feasible allocations. In this paper we focus on a contribution mechanism in which the message (strategy) of each agent is her proposed labor contribution, a point in $[0, \bar{l}]$. The outcome function is such that the labor contribution of each agent is her proposed labor contribution and the consumption is given by a sharing rule. Abusing language we will speak of the mechanism as the sharing rule. The payoff functions induced by the mechanism are $U_i(x_i(l), l_i)$, $i \in N$. A Nash equilibrium of this mechanism is a vector of strategies (l^*) such that

$$U_i(x_i(l^*), l_i^*) \geq U_i(x_i(l_1^*, .., l_i, .., l_n^*), l_i) \text{ for all } l_i \in [0, \bar{l}], \ i \in N.$$

We will refer to (l^*) as a Nash equilibrium associated to the sharing rule $x(\cdot) = (x_1(\cdot), x_2(\cdot), ..., x_n(\cdot))$, or as a Nash equilibrium in short.

Assuming that first order conditions of a Nash equilibrium hold with equality,

$$\frac{\partial U_i(x_i, l_i)}{\partial x_i} \frac{\partial x_i(l)}{\partial l_i} + \frac{\partial U_i(x_i, l_i)}{\partial l_i} = 0, \ i \in N. \qquad (2.3)$$

Let us again remark here that first order conditions may hold with equality at points where $l_i = 0$.

It is clear that a sharing rule may yield efficient allocations as Nash equilibria for some preferences. But it would be desirable to have a sharing rule whose good performance does not depend on being applied to specific preferences. In fact, if preferences can be chosen adequately, efficient outcomes can be achieved not only as Nash equilibria but as equilibria in dominant strategies (see Groves and Loeb [1975] or Liu and Tian [1999]).

Let us define an economy, denoted by $U \equiv (U_1(), U_2(), ..., U_n())$, as a list of utility functions. Let $\mathcal{E}$ be the set of all admissible economies. We assume that the set of admissible economies is large in the following sense:

Assumption D. The set of admissible economies $\mathcal{E}$ contains all economies where utility functions are of the following form:

$$U_i(x_i, l_i) = x_i - \alpha_i l_i - \frac{\beta_i l_i^2}{2},\ \alpha_i,\ \beta_i \geq 0,\ \forall i \in N. \tag{2.4}$$

Assumption D is weaker than the assumption that the domain of possible preferences is the set of all continuous, strictly increasing, and concave utility functions.[4] To explore the implications of Assumption D in the set of efficient allocations we introduce the following notation. Let $\varphi^E(U)$ be the set of efficient allocations in U. Define,

$$R = \{(l \mid \exists U \in \mathcal{E}, (x(l), l) \in \varphi^E(U)\}.$$

In words, R is the set of input allocations that, given a sharing rule $x(\cdot)$, yield efficient allocations for some economy. We refer to l as the efficient input allocation. Define,

$$R_i(l_{-i}) = \{l_i \mid \exists U \in \mathcal{E}, (x(l_i, l_{-i}), (l_i, l_{-i}) \in \varphi^E(U)\}.$$

In words, $R_i(l_{-i})$ is the set of input contributions for i, l_i, such that (l_i, l_{-i}) is an efficient input allocation for some economy.

[4]This assumption is made in the literature characterizing strategy-proof mechanisms, see Moulin (1994, pp. 308-9), Serizawa (1996, pp. 503-8), Osheto (1997, p. 160), Deb and Ohseto (1999, p. 686) and Serizawa (1999, p. 124).

It is easy to see that Assumption D implies the following properties:

$$R_i(l_{-i}) \text{ is the interval } [0, \bar{l}],\ \forall i \in N,\ \forall l_{-i} \in \mathbb{R}_+^{n-1}; \tag{LD}$$

$$\forall l \in [0, \bar{l}]^n, \exists U \in \mathcal{E} \text{ such that } l \text{ is the unique efficient input allocation for } U. \tag{U}$$

For instance, let $U_i(x_i, l_i) = x_i - \beta_i l_i^2/2$ with $\beta_i > 0$. In this case, there is a unique efficient input allocation, say $\hat{l} >> 0$. Moreover any $\hat{l} >> 0$ can be sustained as the unique efficient input allocation of an economy U of the form (2.4) by setting $\alpha_i = 0$ for all $i \in N$, and β's such that

$$\frac{dX(\hat{L})}{dL} = \beta_i \hat{l}_i, \text{ for all } i \in N, \text{ where } \hat{L} = \sum_{i \in N} \hat{l}_i. \tag{2.5}$$

Since the second order conditions of (2.5) hold, the allocation above is efficient. Allocations in which some agents exert zero effort, $\bar{l}$, can be sustained as the unique efficient input allocation of an economy U of the form (2.4) by setting:

$$\begin{aligned} \frac{dX(\bar{L})}{dL} &= \alpha_i,\ \beta_i = 0 \text{ for all } i \in N \text{ such that } \bar{l}_i = 0, \text{ and} \\ \frac{dX(\bar{L})}{dL} &= \beta_j \bar{l}_j,\ \alpha_j = 0 \text{ for all } j \in N \text{ such that } \bar{l}_j \neq 0, \\ \text{where } \bar{L} &= \sum_{i \in N} \bar{l}_i. \end{aligned}$$

Notice that at these boundary input allocations, first order conditions hold with equality.

An important implication of properties LD and U is that if we require that for any U in the domain any Nash equilibrium is efficient, all the input allocations in $[0, \bar{l}]^n$ are Nash equilibria for some U in the domain.

Our first result establishes that only if the production function is a polynomial of, at most, degree $(n-1)$, Nash equilibrium yields efficient allocations for any economy in the domain. The proof is an adaptation of a result from Gradstein (1995) in the framework of the Cournot oligopoly model (a special case of the model considered in this paper).

Theorem 1. Under Assumption D, if Nash equilibrium yields efficient allocations in any $U \in \mathcal{E}$, the production function is a polynomial of, at most, degree $(n-1)$.

Proof. Take any $U \in \mathcal{E}$ and consider a Pareto efficient allocation $(x(l^*), l^*)$ such that l^* is a Nash equilibrium. Thus, from (2.2) and (2.3),

$$\frac{\partial x_i(l^*)}{\partial l_i} = \frac{dX(\sum_{j=1}^n l_j^*)}{dL}, \ \forall i \in N. \tag{2.6}$$

The above equation holds in the interval $R_i(l^*_{-i})$. Integrating on $[0, l_i]$ we get

$$x_i(l_i, l^*_{-i}) = X(l_i + \sum_{j \neq i} l_j^*) - Q_i, \ \forall l_i \in R_i(l^*_{-i}), \ \forall i \in N, \tag{2.7}$$

where Q_i depends on l^*_{-i}. Since the above equation holds for all $l_j \in R_j(l_{-j})$, $\forall j \neq i$,

$$x_i(l) \equiv X(\sum_{j=1}^n l_j) - Q_i(l_{-i}), \ \forall (l_i, l_{-i}) \in R, \ \forall i \in N. \tag{2.8}$$

Adding over i and considering feasibility we obtain

$$(n-1)X(\sum_{j=1}^n l_j) \equiv \sum_{j=1}^n Q_j(l_{-j}), \ \forall l \in R. \tag{2.9}$$

(see Browning (1983)). Consider now all the possible vectors with one component equal to zero. For each of these vectors we apply equation (2.9) and we subtract the resulting equations from equation (2.9). We do the same for all possible vectors with two components equal to zero and we add the equations to the result of the previous step. We proceed in this way subtracting from the previous step the equation resulting from considering all possible vectors with an odd number of components equal to zero and adding the equations resulting from considering all possible vectors with an even number of components equal to zero. As a result of these operations we get the following functional equation:

$$X(L) - \sum_{k=1}^n X(L_{-k}) + \sum_{k<t} X(L_{-kt}) - \sum_{k<t<m} X(L_{-ktm}) + ... + (-1)^{n-1} \sum_{j=1}^n X(l_j) = 0. \tag{2.10}$$

The solution of (2.10) is a polynomial of, at most, degree $(n-1)$ (Aczel (1966) pp. 129-130). ■

An implication of the previous Theorem is that when $n = 2$ and the production function is strictly concave, Nash equilibrium cannot be efficient in all economies in $\mathcal{E}$ because efficiency will require that the production function displays constant returns to scale.

In what follows we show that if the production function is a polynomial of degree, at most, $(n-1)$, there is a contribution mechanism whose Nash equilibria are efficient for all economies in the domain. This mechanism is the one associated to the *incremental sharing rule* formally defined as follows:

Definition 1. *Let the production function be a polynomial of, at most, degree $(n-1)$. The incremental sharing rule is defined as:*

$$x_i^I(l) = X(L) - (n-1)X(L_{-i}) + \frac{n-1}{2}\sum_{k\neq i} X(L_{-ik}) - \frac{n-1}{3}\sum_{k,h\neq i, k<h} X(L_{-ikh}) + .. + (-1)^{n-1}\sum_{j\neq i} X(l_j). \tag{2.11}$$

This sharing rule awards each agent the whole output minus a measure of the contributions of others.

The name *Incremental Sharing Rule* is suggested by the fact that

$$x_i^I(l) - x_i^I(0, l_{-i}) = X(L) - X(L_{-i}). \tag{2.12}$$

The incremental sharing rule, despite the complex analytical form is really simple. It demands the equalization between private gain in consumption of i and public gain in aggregate output for each variation of the labor supplied by i. When $n = 3$ and $n = 4$

the incremental sharing rule looks as follows:

$$x_i^I(l) \;=\; X(L) - 2X(L_{-i}) + \sum_{j\neq i} X(l_j), \text{ and} \qquad (2.13)$$

$$x_i^I(l) \;=\; X(L) - 3X(L_{-i}) + \frac{3}{2}\sum_{k\neq i} X(L_{-ik}) - \sum_{j\neq i} X(l_j). \qquad (2.14)$$

We remark that the incremental sharing rule is well-defined only when $X(\cdot)$ is a polynomial of, at most, degree $(n-1)$. Indeed adding (2.11) over i, we obtain

$$nX(L) + (n-1)\left(-\sum_{k=1}^{n} X(L_{-k}) + \sum_{k<t} X(L_{-kt}) + ... + (-1)^{n-1}\sum_{j=1}^{n} X(l_j)\right). \qquad (2.15)$$

which equals output $X(L)$ when $X(\cdot)$ is a polynomial of, at most, degree $(n-1)$ as shown at the end of Theorem 1.

The next result characterizes the incremental sharing rule as the unique anonymous sharing rule under which Nash equilibrium yields efficient allocations.

Theorem 2. Let $X(\cdot)$ be a polynomial of, at most, degree $(n-1)$. All Nash equilibria associated with an anonymous sharing rule are efficient if and only if the sharing rule is the incremental sharing rule.

Proof. Suppose that Nash equilibrium are efficient. Then, as we have shown in Theorem 1, the sharing rule should satisfy (2.8). Given l, for each agent i, we can find $Q_i(l_{-i})$ applying equation (2.9) successively to $X(L_{-i})$, $X(L_{-ik})$ for all possible k different from i, $X(L_{-ikh})$ for all possible k and h, $k < h$, different from i, and so on up to $X(l_j)$ for all possible j different from i. Given the equations obtained in this way, we apply the following operation:

$$(n-1)X(L_{-i}) - \frac{n-1}{2}\sum_{k\neq i} X(L_{-ik}) - .... - (-1)^{n-1}\frac{n-1}{n-1}\sum_{j\neq i} X(l_j). \qquad (2.16)$$

By anonymity we know that for any vector such that $l_{-i} = l_{-j}$, $Q_i(l_{-i}) = Q_j(l_{-j})$. Thus, applying anonymity to the result of the above operation we get that:

$$Q_i(l_{-i}) = (n-1)X(L_{-i}) - \frac{n-1}{2}\sum_{k\neq i} X(L_{-ik}) - .... - (-1)^{n-1}\sum_{j\neq i} X(l_j). \qquad (2.17)$$

Since $x_i(l) = X(L) - Q_i(l_{-i})$, the sharing rule is the incremental sharing rule as we wanted to prove.

Consider now the existence of a Nash equilibrium associated with the incremental sharing rule. This is proved by noting that, under the conditions stated below, each agent, say i, maximizes a concave function over the incremental sharing rule that is concave in the labor supplied by i. Thus the Best Reply is convex-valued and upper-hemi-continuous. Thus a standard fixed point argument shows the existence of a Nash equilibrium. Finally, notice that the allocation yielded by Nash equilibrium is efficient because the incremental sharing rule equalizes social and private gains as in equation (2.6). Clearly, this sharing rule is anonymous. ■

Finally, we investigate the implications of relaxing anonymity. We assume instead that the sharing rule can be written as a function of the sum of the contributions or it yields zero consumption when the corresponding labor contribution is zero. Most of sharing rules considered in the literature fulfill, at least, one of these two properties (see, e.g. Moulin [1987], Pfingsten [1991] and Roemer and Silvestre [1993]).

Proposition 1. *Assume that the sharing rule can be written as $x_i = x_i(l_i, \sum_{j=1}^n l_j)$, or it is such that $x_i(0, l_{-i}) = 0$. Under Assumption D, if Nash equilibrium yields efficient allocations in any $U \in \mathcal{E}$, the production function displays constant returns to scale.*

Proof. **Case 1**. Let us consider first the case $x_i = x_i(l_i, \sum_{j=1}^n l_j)$. Since the production function depends on the sum of inputs, $Q_i(0, l_{-i}) = Q_i(\sum_{j \neq i} l_j)$. For any possible vector l such that $\sum_{j=1}^n l_j \leq \bar{l}$, consider another vector such that all components are zero except one (let us say i) and this component is the sum of all components in l. Then, equation (2.9) implies that:

$$(n-1)X(L) = \sum_{j \neq i} Q_j(L). \tag{2.18}$$

Repeating the argument but considering that the non-zero component is k,

$$(n-1)X(L) = \sum_{j \neq k} Q_j(L). \tag{2.19}$$

Subtracting both equations we get that

$$Q_i(L) = Q_k(L). \tag{2.20}$$

Since the above equation is true for any i and k, from (2.18),

$$(n-1)X(L) = (n-1)Q_k(L), \tag{2.21}$$

which implies $Q_k(L) = X(L)$ for all k and L. Given l such that $\sum_{j=1}^{n} l_j \leq \bar{l}$, consider another vector with the first component equal to l_1, the second component equal to $\sum_{j \neq 1} l_j$ and any other component equal to zero. Equation (2.9) now reads

$$(n-1)X(\sum_{j=1}^{n} l_j) = X(\sum_{j \neq 1} l_j) + X(l_1) + (n-2)X(\sum_{j=1}^{n} l_j), \tag{2.22}$$

which implies that

$$X(\sum_{j=1}^{n} l_j) = X(\sum_{j \neq 1} l_j) + X(l_1). \tag{2.23}$$

Repeating the argument to $X(\sum_{j \neq 1} l_j)$ and so on, we get that

$$X(\sum_{j=1}^{n} l_j) = \sum_{j=1}^{n} X(l_j), \text{ for all } l \text{ such that} \sum_{j=1}^{n} l_j \leq \bar{l}. \tag{2.24}$$

This is a Cauchy equation whose solutions are linear (Aczel (1966), chapter 2). Thus, the production function displays constant returns to scale for all l such that $\sum_{j=1}^{n} l_j \leq \bar{l}$. By Theorem 1 the production function is a polynomial of degree $n-1$. Combining both results, the production function displays constant returns to scale in the whole domain.

Case 2. If the sharing rule is such that $x_i(0, l_{-i}) = 0$, we have that $x_i(0, l_{-i}) = 0 = X(L_{-i}) - Q_i(l_{-i})$. Thus, $Q_i(l_{-i}) = X(L_{-i})$ for all i and (2.9) reads:

$$(n-1)X(\sum_{j=1}^{n} l_j) \equiv \sum_{j=1}^{n} X(L_{-j}). \tag{2.25}$$

Let us see that equation (2.25) implies that

$$X(\sum_{j=1}^{n} l_j) = \sum_{j=1}^{n} X(l_j), \tag{2.26}$$

which is a Cauchy's equation whose solutions are linear (see Aczel 1966, chapter 2 for a discussion). We prove the above relation by induction on the number of zero components in a vector l. Let us consider a vector l such that all components but two are zero. Thus,

$$\begin{aligned}(n-1)X(l_i + l_j) &= X(l_i) + X(l_j) + (n-2)X(l_i + l_j), \text{ thus} \\ X(l_i + l_j) &= X(l_i) + X(l_j).\end{aligned}$$

Suppose that the relation is true for all vectors l such that all components but one are different from zero. Then, applying the induction hypothesis to equation (2.25),

$$(n-1)X(\sum_{j=1}^{n} l_j) = \sum_{j=1}^{n} X(l_{-j}) = (n-1)\sum_{j=1}^{n} X(l_j),$$

as we wanted to prove. ■

Proposition 1 implies that the result obtained by Sen is an artifact of his assumption that all agents are identical. In his case our Assumption D fails because $R_i(l_{-i})$ is just a point or the empty set. We notice that under constant returns to scale, there is a dominant strategy contribution mechanism achieving efficiency, namely $x_i = l_i$ (wlog we assumed that $X(L) = L$). Thus, under constant returns, implementation of efficient allocations by means of a contribution mechanism is possible in a more robust equilibrium concept, namely dominant strategies. This finding is not robust, though, because it does not hold under decreasing returns to scale, see Leroux (2004). When the inputs are owned collectively, the problem of implementing in dominant strategies efficient allocations under constant returns to scale is far from trivial, see Maniquet and Sprumont (1999).

3. Final Remarks

In this paper we have studied the problem of finding sharing rules whose Nash equilibrium yield efficient allocations. We found that this is only possible when the production function is a polynomial of, at most, degree $n-1$. We also show that the only anonymous sharing rule doing this job is the incremental sharing rule. Finally, we have shown that, for a large class of sharing rules, efficiency and Nash equilibrium are only compatible under constant returns to scale, a case in which efficient allocations can be achieved by a strategy-proof mechanism.

To end up the paper, we remark here some properties of the incremental sharing rule.

Firstly we recall that the consumption yielded by Sen´ sharing rule is between those yielded by the proportional and the egalitarian sharing rules. This property does not hold for the incremental sharing rule. However it is very easy to see that a related property holds for this sharing rule, namely, that the *increase* in consumption resulting from an increase in i´s labor in the incremental sharing rule (2.11) is between those yielded by the proportional and the egalitarian sharing rules.

Secondly, it can be shown that, in some cases, the incremental sharing rule yields nonnegative shares: For instance if $n = 2$ because in this case the production function is linear. When $n > 2$ this is also the case when the production function is an increasing and concave polynomial whose unique positive coefficient is the corresponding to L, or when $n > 3$ and all but the coefficient of L and L^2 are positive. The proofs of these two last results are delegated to the Appendix.

4. Appendix.

Lemma 1. *Let*

$$S_n = -(n-2) + \sum_{k=m}^{n-2} (-1)^{n-k} \frac{(n-1)}{n-k} \binom{n-m-1}{k-m}.$$

Then, $S_n = \frac{-m+1}{n-m}$.

Proof. Notice first that

$$\begin{aligned} \frac{1}{n-k}\binom{n-m-1}{k-m} &= \frac{1}{n-m}\binom{n-m}{n-k}, \text{ and} \\ \sum_{k=m}^{n-2} (-1)^{n-k}\binom{n-m}{n-k} &= \sum_{k=2}^{n-m} (-1)^{k}\binom{n-m}{k}. \end{aligned}$$

Thus,

$$S_n = -(n-2) + \frac{(n-1)}{(n-m)} \sum_{k=2}^{n-m} (-1)^{k}\binom{n-m}{k}.$$

By the Newton's binomial we know that

$$0 = (1+(-1))^{n-m} = \sum_{k=0}^{n-m} (-1)^{k}\binom{n-m}{k}.$$

Thus,

$$S_n = -(n-2) + \frac{(n-1)}{(n-m)}(-1+n-m) = \frac{-m+1}{n-m},$$

as we wanted to prove. ■

Proposition 2. *Let* $n \geq 3$, *and let* $X(L) = a_{n-1}L^{n-1} + a_{n-2}L^{n-2} + ... + a_2L^2 + a_1L$ *an increasing and concave polynomial in* $[0, n\bar{l}]$ *with* $a_t \leq 0$ *for all* $t \in \{2, ..., n-1\}$. *Then,* $x_i^I(l) \geq 0$ *for all* i.

Proof. Since $x_i^I(l)$ is increasing in l_i (recall that $x_i^I(l) = X(L) - Q_i(l_{-i})$), to prove the proposition it is enough to show that $x_i^I(0, l_{-i}) \geq 0$. Without lost of generality let us consider $i = n$. From (2.11) we know that

$$x_n^I(0, l_{-n}) = -(n-2)X(L_{-n}) + \frac{n-1}{2}\sum_{k=1}^{n-1} X(L_{-nk}) + ... + (-1)^{n-1}\sum_{j=1}^{n-1} X(l_j). \quad (4.1)$$

Let $X_t(L) = a_t L^t$, $1 \leq t \leq n-1$, and let us show that for all $t \in \{1, .., n-1\}$,

$$-(n-2)X_t(L_{-n}) + \frac{n-1}{2}\sum_{k=1}^{n-1} X_t(L_{-nk}) + ... + (-1)^{n-1}\sum_{j=1}^{n-1} X_t(l_j) \geq 0. \quad (4.2)$$

Notice first that, for a given set of m components, $1 \leq m \leq n-1$, without loss of generality, let us call them $l_1, ..., l_m$,

$$X_t(l_1 + .. + l_m) = a_t \sum_{t_1,..,t_m} \frac{t!}{t_1!...t_m!} l_1^{t_1} l_2^{t_2} ... l_m^{t_m}, \quad (4.3)$$

where the sum is taken over all non negative integers $t_1, .., t_m$ such that $t_1 + ... + t_m = t$. Thus, expression (4.2) can be rewritten as an expression with terms of the form $l_1^{t_1} l_2^{t_2} ... l_m^{t_m}$ with $1 \leq m \leq t$, $t_h > 0$ for all $h \in \{1, ..., m\}$ and $t_1 + ... + t_m = t$. Let us see that all the coefficients of such terms are positive. Fix m, and $t_1, ..., t_m$ all positive and such that $t_1 + ... + t_m = t$. For each of the terms involving the sum of k components ($k \geq m$) in expression (4.2), $l_1^{t_1} l_2^{t_2} ... l_m^{t_m}$ appears as many times as the number of combinations of $(n-m-1)$ elements taken $(k-m)$ at a time. Thus, the coefficient of $l_1^{t_1} l_2^{t_2} ... l_m^{t_m}$ is:

$$a_t \frac{t!}{t_1!...t_m!} \left[-(n-2) + \sum_{k=m}^{n-2} (-1)^{n-k} \frac{(n-1)}{(n-k)} \binom{n-m-1}{k-m} \right]. \quad (4.4)$$

As we have proved in Lemma 1,

$$-(n-2) + \sum_{k=m}^{n-2} (-1)^{n-k} \frac{(n-1)}{n-k} \binom{n-m-1}{k-m} = \frac{-m+1}{n-m}. \quad (4.5)$$

Since $a_t \leq 0$ for all $t \in \{2, ..., n-1\}$ the coefficient of $l_1^{t_1} l_2^{t_2} ... l_m^{t_m}$ is non negative. Therefore expression (4.2) is non negative for all $t \in \{2, .., n-1\}$. For $t = 1$, a_1 is positive and sufficiently large to guarantee that the polynomial is increasing, but in this case we only have terms of the form t_j, and all the coefficients of these terms are zero (since $m = 1$). Thus, expression (4.2) is non negative for all t as we wanted to prove. ■

Lemma 2. *For all* t, $3 \leq t \leq n-1$, *and for all* m, $2 \leq m \leq n-1$,

$$\frac{-m+1}{n-m} + \frac{1}{(n-2)}(\frac{n}{n-1})^{t-2}\frac{m(m-1)}{2} \geq 0. \tag{4.6}$$

Proof. Notice first that in order to prove the statement it is enough to show that for all $m, 2 \leq m \leq t$,

$$-1 + \frac{1}{(n-2)}(\frac{n}{n-1})^{t-2}\frac{m(n-m)}{2} \geq 0. \tag{4.7}$$

Clearly, for $m = 2$, the statement is true. Notice that $m(n-m)$ as a function of m extended to the real numbers is concave in m,thus, the minimum of this function is reached in $m = 2$ or $m = t$. Notice that $m(n-m)$ gets the same value for $m = 2$ and for $m = n-2$. Therefore, if $t \leq n-2$, for all $2 \leq m \leq t$, $m(n-m) \geq 2(n-2)$. Thus, the expression in (4.7) is positive. If $t = n-1$, the minimum is obtained in $m = t$. Let us show that also in this case the statement of the Lemma holds, that is:

$$-1 + \frac{1}{(n-2)}(\frac{n}{n-1})^{n-3}\frac{(n-1)}{2} \geq 0, \text{ or equivalently,} \tag{4.8}$$

$$\frac{(n-1)}{(n-2)}(\frac{n}{n-1})^{n-3} \geq 2. \tag{4.9}$$

Let $a_n = \frac{(n-1)}{(n-2)}(\frac{n}{n-1})^{n-3}$, $b_n = \frac{(n-1)}{(n-2)}\left(\frac{n-1}{n}\right)^2$ and $c_n = (\frac{n}{n-1})^{n-1}$. Notice that $a_n = b_n c_n$. The sequence c_n is a well studied sequence which is increasing and converges to the number e, and it is easy to prove that the sequence b_n is increasing in n for all $n \geq 4$. Thus, a_n is increasing in n and since $a_4 = 2$, $a_n \geq 2$ for all $n \geq 4$. ■

Proposition 3. *Let* $n \geq 4$, *and let* $X(L) = a_{n-1}L^{n-1} + a_{n-2}L^{n-2} + ... + a_2L^2 + a_1L$ *an increasing and concave polynomial in* $[0, n\bar{l}]$ *with* $a_t \geq 0$ *for all* $t \in \{3, ..., n-1\}$. *Then,* $x_i^I(l) \geq 0$ *for all* i.

Proof. Notice first that the proof of Proposition 2 can be reproduced here up to (4.5). From expressions (4.4), and (4.5) we know that for a given t, $1 \leq t \leq n-1$, for a fixed m, $1 \leq m \leq t$, and for $t_1, ..., t_m$ all positive and such that $t_1 + ... + t_m = t$, the coefficient of terms of the form $l_1^{t_1} l_2^{t_2} ... l_m^{t_m}$ is given by

$$a_t \frac{t!}{t_1!...t_m!}(\frac{-m+1}{n-m}). \tag{4.10}$$

For $m = 1$, the coefficient is zero. So we just fix attention to $m \geq 2$. For $t = 2$, we only have terms of the form $l_j l_k$ with coefficient $-\frac{2}{(n-2)} a_2$. Thus, expression (4.2) is non positive for all $t \in \{3, .., n-1\}$ since $a_t \geq 0$, it is positive for $t = 2$ since $a_2 < 0$, and it is cero for $t = 1$. However, let us see that (4.1) is positive. By concavity of the polynomial,

$$2a_2 \leq -\sum_{t=3}^{n-1} t(t-1) a_t L^{t-2} \text{ for all } L \in [0, n\bar{l}]. \tag{4.11}$$

In particular, (4.11) holds for $L = \frac{n}{n-1}(\sum_{j=1}^{n-1} l_j)$. Thus, for all possible combinations of $l_j l_k$,

$$-\frac{2}{(n-2)} a_2 l_j l_k \geq \frac{1}{(n-2)} \sum_{t=3}^{n-1} t(t-1) a_t (\frac{n}{n-1})^{t-2} l_j l_k (\sum_{j=1}^{n-1} l_j)^{t-2}. \tag{4.12}$$

For a given t, $3 \leq t \leq n-1$, for a fixed m, $2 \leq m \leq t$, and for $t_1, ..., t_m$ all positive and such that $t_1 + ... + t_m = t$, the term $l_1^{t_1} l_2^{t_2} ... l_m^{t_m}$ appears in all the inequalities in (4.12) involving all possible order pairs $l_j l_k$ among $(l_1, ..., l_m)$. Thus, the coefficient of $l_1^{t_1} l_2^{t_2} ... l_m^{t_m}$ that is obtained from those inequalities is:

$$\sum_{T_{jk}^m} \frac{1}{(n-2)} (\frac{n}{n-1})^{t-2} t(t-1) a_t \frac{(t-2)!}{t_1!..(t_j-1)!..(t_k-1)!...t_m!}, \tag{4.13}$$

where the sum is taken over all possible pairs of indexes in the set $T_{jk}^m = \{(j,k)/j < k,$ and $j, k \in \{1, ..., m\}\}$. Notice that expression (4.13) can be rewritten as:

$$\sum_{T_{jk}^m} \frac{1}{(n-2)} (\frac{n}{n-1})^{t-2} a_t \frac{t! t_j t_k}{t_1!..t_j!..t_k!...t_m!}. \tag{4.14}$$

Since $t_j t_k \geq 1$, and the cardinality of the set T^m_{jk} is equal to the combinations of m elements taken two at a time, expression (4.14) is bigger than

$$\frac{1}{(n-2)}(\frac{n}{n-1})^{t-2} a_t \frac{t!}{t_1!..t_j!..t_k!...t_m!} \frac{m(m-1)}{2}. \tag{4.15}$$

Thus, combining (4.10) and (4.15), we get that the coefficient of $l_1^{t_1} l_2^{t_2} ... l_m^{t_m}$ is

$$a_t \frac{t!}{t_1!...t_m!} \left[\frac{-m+1}{n-m} + \frac{1}{(n-2)}(\frac{n}{n-1})^{t-2} \frac{m(m-1)}{2} \right]. \tag{4.16}$$

As we have shown in Lemma 2, (4.16) is positive for all m, $2 \leq m \leq t$, $3 \leq t \leq n-1$, which implies that (4.1) is positive as we wanted to prove. ■

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