GAME THEORY
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Problem Set 1
(to be returned on Friday, September 19th)

EXERCISE 1. (Contributing to a public good) Each of n people chooses whether or not to contribute a fixed amount toward the provision of a public good. The good is provided if and only if at least k people contribute, where $2 < k < n$; if it is not provided, contributions are not refunded. Each person ranks outcomes from best to worst as follows: (i) any outcome in which the good is provided and she does not contribute, (ii) any outcome in which the good is provided and she contributes, (iii) any outcome in which the good is not provided and she does not contribute, (iv) any outcome in which the good is not provided and she contributes.

Formulate this situation as a strategic game and find its Nash equilibria. (Is there a Nash equilibrium in which more than k people contribute? One in which k people contribute? One in which fewer than k people contribute?)

EXERCISE 2. (Games without conflict) Give some examples of two-player strategic games in which each player has two actions and the players have the same preferences, so that there is no conflict between their interests.

EXERCISE 3. (Voter participation) Two candidates, A and B, compete in an election. Of the n citizens, k support candidate A and m $(= n - k)$ support candidate B. Each citizen decides whether to vote, at a cost, for the candidate she supports, or to abstain. A citizen who abstains receives the payoff of 2 if the candidate she supports wins, 1 if this candidate ties for first place, and 0 if this candidate loses. A citizen who votes receives the payoffs $2 - c$, $1 - c$, and $-c$ in these three cases, where $0 < c < 1$.

(a) For $k = m$, find the set of Nash equilibria. (Is the action profile in which everyone votes a Nash equilibrium? Is there any Nash equilibrium in which the candidates tie and not everyone votes? Is there any Nash equilibrium in which one of the candidates wins by one vote? Is there any Nash equilibrium in which one of the candidates wins by two or more votes?)

(c) What is the set of Nash equilibria for $k < m$?

EXERCISE 4. Let $(N, (S_i)_{i\in N}, (u_i)_{i\in N})$ and $(N, (S_i)_{i\in N}, (u'_i)_{i\in N})$ two complete information games in strategic form where, for every player $i\in N$, the payoff function satisfies $u'_i = \alpha_i + \beta_i\, u_i$ and α_i and β_i are two real numbers, $\beta_i > 0$. Prove that the set of Nash equilibria of the (mixed extension) of the two games coincide; that is, positive affine transformations of the payoffs functions do not change the set of Nash equilibria. Comment on the importance of this result.

EXERCISE 5. (A joint project) Two people are engaged in a joint project. If each person i puts in the effort $x_i \in [0, 1]$ at a cost of $c(x_i)$ to her, the outcome of the project is worth $f(x_1, x_2)$. The worth of the project is split equally between the two people, regardless of their effort levels. Formulate this situation as a strategic game. Find the Nash equilibria of the game when:

(a) $f(x_1, x_2) = 3x_1x_2$ and $c(x_i) = x_i^{\,2}$ for $i = 1, 2$, and

(b) $f(x_1, x_2) = 4x_1x_2$ and $c(x_i) = x_i$ for $i = 1\ 2$.

In each case, is there a pair of effort levels that yields both players higher payoffs than the Nash equilibrium effort levels?

EXERCISE 6. (Contributing to a public good) Consider a situation in which two people decide how much to contribute to a public good. A person i's wealth is denoted w_i and her contribution by c_i. Person i's utility $u_i(c_1 + c_2, w_i - c_i)$ is the sum of three parts: the amount $c_1 + c_2$ of the public good provided, the amount $w_i - c_i$ person i spends on private goods, and a term $(w_i - c_i)(c_1 + c_2)$ that reflects an interaction between the amount of the public good and her private consumption (the greater the amount of the public good, the more she values her private consumption). In summary, suppose that person i's payoff is $c_1 + c_2 + w_i - c_i + (w_i - c_i)(c_1 + c_2)$, or

$$\mathrm{w_i + c_j + (w_i - c_i)(c_1 + c_2)}$$

where j is the other person. Assume that $w_1 = w_2 = w$, and that each player i's contribution c_i may be any number (positive or negative, possibly larger than w)

(a) Find the Nash equilibrium of the game that models this situation. (You can calculate the best responses explicitly. Imposing the sensible restriction that c_i lie between 0 and w complicates the analysis, but does not change the answer.) Show that in the Nash equilibrium both players are worse off than they are when they both contribute one half of their wealth to the public good.

(b) Extend the analysis to the case of n people. As the number of people increases, how does the total amount contributed in a Nash equilibrium change? Compare the players' equilibrium payoffs with their payoffs when each contributes half her wealth to the public good, as n increases without bound.

EXERCISE 7. (First-price sealed-bid auction with two bidders) Find *all* the Nash equilibria of a first-price sealed-bid auction with two bidders, where $v_1 > v_2$. (We *always* keep the convention that in case of equal bids, the object goes to the lowest numbered bidder.)

Extend your arguments and find all the Nash equilibria for the case of n bidders, where the valuations are $v_1 > v_2 > ... > v_n > 0$.

EXERCISE 8. (Third-price auction) Consider a *third*-price sealed-bid auction which differs from a first- and a second-price auction only in that the winner (the person who submits the highest bid) pays the third highest price. (Assume that there are at least three bidders.)

(a) Show that for any player i the bid of v_i weakly dominates any lower bid, but does not weakly dominate any higher bid.

(b) Show that the action profile in which each player bids her valuation is not a Nash equilibrium.

(c) Find a Nash equilibrium. (There are ones in which every player submits the same bid.)

EXERCISE 9. (Waiting in line) Two hundred people are willing to wait in line to see a movie at a theater whose capacity is one hundred. Denote person i's valuation of the movie in excess of the price of admission, expressed in terms of the amount of time she is willing to wait, by v_i. That is, person i's payoff if she waits for t_i units of time is $v_i - t_i$. Each person attaches no value to a second ticket, and cannot buy tickets for other people. Assume that $v_1 > v_2 > ... > v_{200}$. Each person chooses an arrival time. If several people arrive at the same time then their order in line is determined by their index (lower-numbered people go first). If a person arrives to find 100 or more people already in line, her payoff is zero. Model the situation as a variant of a discriminatory multi-unit auction, in which each person submits a bid for only one unit, and find its Nash equilibria. (Look at your answer to Exercise 7 before seeking the Nash equilibria.) Arrival times for people at movies do not in general seem to conform with a Nash equilibrium. What feature missing from the model could explain the pattern of arrivals?

EXERCISE 10. Find all the mixed strategy Nash equilibria of the two-player strategic games:

	L	R
T	6,0	0 *6*
B	3,2	6,0

	L	R
T	0,1	0,2
B	2,2	0,1

EXERCISE 11. (Choosing numbers) Players 1 and 2 each choose a positive integer up to K. If the players choose the same number then player 2 pays $1 to player 1; otherwise no payment is made. Each player's preferences are represented by her expected monetary payoff.

Find the unique mixed strategy Nash equilibrium of the game.

EXERCISE 12. (Finding all mixed strategy equilibria of a two-player game) Find all the mixed strategy Nash equilibria of the following strategic game:

	L	M	R
T	2,2	0,3	1,3
B	3,2	1,1	0,2