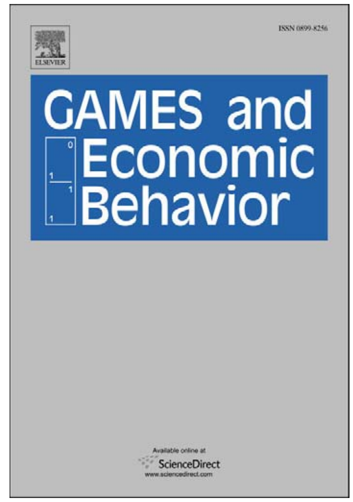
Accepted Manuscript



Diffusion in Complex Social Networks

Dunia López-Pintado

PII: S0899-8256(07)00115-7
DOI: 10.1016/j.geb.2007.08.001
Reference: YGAME 1416

To appear in: *Games and Economic Behavior*

Received date: 27 January 2007
Revised date: 30 July 2007
Accepted date: 9 August 2007

Please cite this article as: D. López-Pintado, Diffusion in Complex Social Networks, *Games and Economic Behavior* (2007), doi: 10.1016/j.geb.2007.08.001

Diffusion in Complex Social Networks*

Dunia López-Pintado
Universitat Autònoma de Barcelona

July 30, 2007

Address for correspondence:
Departament d'Economia i d'Història Econòmica
Universitat Autònoma de Barcelona
Edifici B, 08193 Bellaterra, Spain.
E-mail: dunia.lopez@uab.es
Tlf: 935814254. Fax: 935812012.

*This paper is based on Chapter 1 of my Ph.D. thesis at the University of Alicante. I am most grateful to my advisor Fernando Vega-Redondo for his continuous encouragement and support. Thanks are also due to Sudeep Bisla, Scott Boorman, Antoni Calvó-Armengol, Leon Danon, Peter Dodds, Matthew Jackson, Ricardo Martinez, Juan Moreno-Ternero, Stephen Morris and Duncan Watts for valuable comments and suggestions. I also acknowledge the participants' contributions at seminars and conferences where the paper has been presented, at Alicante, Caltech, Columbia, Fordham, Lisbon, Osaka, Princeton, San Sebastián and the NBER/NSF 2005 Decentralization Conference (Urbana-Champaign). Financial support from the IVIE and the James S. McDonnell Foundation in form of a post-doctoral fellowship is gratefully acknowledged. Financial support from the Spanish Ministry of Education and Science through grant SEJ2006-27589-E, FEDER and Barcelona Economics-XREA is gratefully acknowledged.

Abstract

This paper studies how a behavior spreads in a population. We consider a network of interacting agents whose actions are determined by the actions of their neighbors, according to a simple diffusion rule. We find, using a *mean-field* approach, the threshold for the spreading rate above which the behavior spreads and becomes persistent in the population. This threshold crucially depends on the connectivity distribution of the social network and on specific features of the diffusion rule.

Keywords: social networks, diffusion, mean-field theory, connectivity, hysteresis

JEL Classification Numbers: C73, L14, O31, O33.

1 Introduction

Individuals decisions are often influenced by the decisions of other individuals. Regardless of the origins of the influence process (whether it is due to information transmission, coordination effects or a mixture of both) there is a wide range of social phenomena such as diffusion of innovations, cultural fads, local variability in crime activities, or economic conventions that share the logic of contagion (e.g., Aguirre et al., 1988; Glaeser et al., 1996; Rogers, 1995; Young and Burke, 2001).

We analyze this phenomenon in a stylized model of social networks. The social network is characterized by a set of individuals (nodes) and a set of relationships across these individuals (links) through which social influence operates. We study a dynamic model in which agents choose whether to adopt or not the new behavior in each period as a function of the actions taken by their neighbors in the previous period. We consider a continuous mean-field version of this dynamics which assumes that the network is realized every period and thus neighbors are a random sample from the population. Therefore, the main property of the network is its connectivity distribution (where the connectivity of an agent is the number of direct neighbors she has). We show how the diffusion in the population of the new behavior depends on the connectivity distribution of the social network as well as on the diffusion (or contagion) mechanism.

There exists a recent literature concerning the issue of diffusion on networks (see Jackson, 2006; Vega-Redondo, 2006; and the literature cited therein). This literature initially focussed on networks characterized by a recurrent pattern (e.g., Blume, 1995; Ellison, 1993). Later, the interest shifted towards diffusion on general network structures, increasing the complexity of the problem (e.g., Dodds and Watts, 2004; Jackson and Rogers, 2007a; Morris, 2000; Pastor-Satorrás and Vespignani, 2001; Watts, 2002; Young, 1998).

This paper builds on the so-called Susceptible-Infected-Susceptible (SIS) model (e.g., Pastor-Satorrás and Vespignani, 2001). The SIS model, frequently used in epidemiology, considers a specific diffusion rule that is linear on the absolute number of infected neighbors. This implies, for example, that an agent with 2 infected neighbors has a probability of becoming infected which is independent on the size of her neighborhood (i.e., only absolute numbers matter). In addition, an agent with 4 infected neighbors has twice the chances of becoming infected than an agent with 2 infected neighbors. The main contribution of Pastor-Satorrás and Vespignani (2001) is to identify conditions under which a population is susceptible to an epidemic by means of a mean-field approach. In this paper, we also take the mean-field approach to analyze diffusion in a wide range of social and economic contexts. We consider general diffusion rules which include not only the SIS model, but also imitation models (in which individuals imitate the behavior of a neighbor), threshold models (in which individuals only adopt if a certain fraction of their neighbors adopt), etc. The SIS model has also been recently studied by Jackson and Rogers (2007a), who analyze the diffusion properties of networks ordered through the stochastic dominance of their connectivity distributions. Some of their techniques are also used here to order diffusion thresholds in our different models.

In this paper, we provide a closed-form solution for the diffusion threshold of the spreading rate of the new behavior in a given network. The value of the diffusion threshold crucially depends on two components: the diffusion function and the connectivity distribution of the network. We show, for instance, that if the diffusion function is such that only the absolute number of adopters matters, the higher the variance of the connectivity distribution the smaller the diffusion threshold. For other diffusion functions in which the size of the neighborhood plays a role, the opposite might happen. Furthermore, we show that, for concave diffusion functions, there exists a unique limit state of the contagion process. However, *hysteresis* can occur for non-concave diffusion functions.

The rest of the paper is organized as follows. In Section 2 we introduce the model. In Section 3 we present the main results. Section 4 concludes providing further connections with the related literature. Some of the proofs have been relegated to an appendix.

2 The Model

2.1 The Network

Consider a finite (but large) set of agents $N = \{1, 2, ..., i, ...n\}$. Each agent $i \in N$ interacts with a subset of the population which determines a social network. Let $\Gamma \equiv (N, L)$ be the social network, where $L \subseteq N \times N$ is the set of pairwise interactions among the individuals in the population (i.e., the links). We consider undirected networks, i.e., if $(i, j) \in L$ then $(j, i) \in L$. We rule out the existence of reflexive links, i.e., $(i, i) \notin L$ for any given $i \in N$. In addition, let N_i be the set of individuals with whom i is linked. Formally,

$$N_i = \{j \in N, \text{ such that } (i, j) \in L\}$$

where $k_i \equiv |N_i|$ is the number of neighbors of i, often referred as her connectivity (or degree).

As we shall explain later, the key property of the network will be its connectivity distribution. For any network Γ, the connectivity distribution $P(k)$ displays for each $k \geq 1$ the fraction of nodes with connectivity k in the population, where $\sum_{k\geq 1} P(k) = 1$.[1]

The following three examples of networks play a prominent role in our discussion. A *homogeneous* network is such that all nodes have the same connectivity (i.e., $P(\overline{k}) = 1$, for some $\overline{k} \geq 1$). An *exponential* network is such that the connectivity distribution is an

[1] We assume that all agents have at least one neighbor, and therefore $P(0) = 0$.

exponential function (i.e., $P(k) \propto e^{-k}$, where "$\propto$" stands for "equal up to a multiplicative constant"). In this case, although there is heterogeneity, the average connectivity is representative of the connectivity of most individuals in the population. Finally, a *scale-free* network exhibits a power-law connectivity distribution (i.e., $P(k) \propto k^{-\gamma}$, where γ ranges between 2 and 3) which implies that there is a large variance in the connectivity of nodes. The interest in the study of scale-free networks is enhanced by the empirical evidence that many paradigmatic examples of complex networks such as the WWW, the Internet and the human sexual contact networks, among others, are characterized by scale-free connectivity distributions (e.g., Albert et al., 1999; Lijeros et al., 2001).

2.2 The Diffusion Mechanism

Our model studies diffusion of a new behavior in a population. To be more precise we concentrate on the example of diffusion of innovations. Assume there is a new product on the market. We focus on its spreading among the population N and its persistence. To do so, we consider that an agent $i \in N$ can only exist in two discrete states: $s_i \in \{0,1\}$, where $s_i = 0$ if i is a "susceptible" agent, and $s_i = 1$ if i is an "active" agent. A susceptible agent is one who does not have the product but is susceptible to obtaining it if exposed to someone who does. An active agent is one who has already adopted the product and so can influence her neighbors in favor of adopting it. Consider the following continuous dynamic process. At time t, the state of the system is a vector

$$s_t = (s_{1t}, s_{2t}, ...s_{it}, ...s_{nt}) \in S^n \equiv \{0,1\}^n ,$$

where $s_{it} = 0$ if i is a susceptible agent at time t, whereas $s_{it} = 1$ if i is an active agent at time t. Let i be a susceptible agent at time t. Then i becomes active at a rate $F(\nu, k_i, a_i)$ which depends on: the spreading rate (or degree of contagion) of the product $\nu \geq 0$, her connectivity k_i, and the number of neighbors who are active at time t ($a_i \equiv \sum_{j \in N_i} s_j$).[2] We assume that the spreading rate effect and the neighbors effect are independent. Formally,

$$F(\nu, k_i, a_i) = \nu \cdot f(k_i, a_i), \tag{1}$$

where the *diffusion function* $f(k_i, a_i)$ is a non-negative function defined for $(k_i, a_i) \in N \times N$ such that $0 \leq a_i \leq k_i$ and non-decreasing as a function of a. It is worth noting that the connectivity of an agent is fixed throughout the dynamics. On the other hand, the number of active agents among neighbors (a_i) might change over time.

One plausible interpretation for the transition from susceptible to active is the following. At a rate ν any given agent becomes aware of the existence of the product (e.g., through

[2] We define rates instead of probabilities because we consider a continuous dynamics. The intuition should be that in a small increment of time dt, the probability of adopting the product is $F(\nu, k_i, a_i)dt$.

mass-media advertisement) and considers the possibility of adopting it. The agent's final decision, however, depends critically on her neighbors' behavior. More precisely, the agent responds to her neighbors' current configuration by choosing an action according to some choice rule characterized by the diffusion function $f(k_i, a_i)$.

The reverse transition (from active to susceptible) is also possible. We assume that an active agent becomes susceptible at some stochastically constant rate $\delta > 0$, which is independent of the state of the neighborhood. It is implicit in this formulation that the cost of "maintaining" the product is approximately zero and thus agents never have incentives to get rid of it. At a rate δ, however, an agent may need to replace the product because it is lost or deteriorated.[3]

Finally, let us define the *effective spreading rate* of the product by $\lambda = \frac{\nu}{\delta}$. For concreteness, we will now determine formally what we mean by the diffusion mechanism.

A ***diffusion mechanism*** *is a pair* $m = (\lambda, f(\cdot))$ *where* λ *denotes the effective spreading rate of the product and* $f(\cdot)$ *denotes the diffusion function.*

Notice that, since the transition rates only depend on the properties of the present state, the dynamics, induced by the connectivity distribution $P(k)$ and the diffusion mechanism m, determines a continuous *Markov chain* over the space of possible states S^n. The analytical results of this dynamics are extremely complicated and thus will not be addressed. We concentrate instead on the study of a *mean-field* dynamics described below.

2.3 The Mean-Field Dynamics

There are several assumptions implicit in a mean-field approach.

First, we assume that the network of interaction is realized every period. In other words, at each time t agents choose their neighbors randomly from the population. In such a case, the mean-field dynamics simplifies the problem in a way so that we are able to compare network structures in terms of the properties of their connectivity distributions.

Second, the stochastic dynamics is substituted by a deterministic dynamics. This approximation is appropriate when dealing with large populations as described in Benaim and Weibull (2003). They show that, if the deterministic population flow remains forever in some subset of the state space, then the stochastic process will remain in the same subset space with a probability arbitrarily close to one, provided that the population is large enough. Consequently, in what follows, we shall assume that the population is infinite.

[3] It is relatively straightforward to modify this model and assume that the transition from active to susceptible also depends on the behavior of neighbors. This alternative case is explored in López-Pintado (2006, 2007) and Watts (2002) for specific diffusion functions and, more generally, in Jackson and Yariv (2006). In particular, these papers assume that an agent's choice in a certain period only depends on the neighborhood's configuration and thus is independent on whether the agent is currently active or susceptible.

The mean-field approximation allows us to address questions that otherwise would be intractable. For instance, given a certain diffusion function, how does the connectivity distribution of the social network affect the spreading pattern of the product? Furthermore, given a certain network (characterized by its connectivity distribution), how does the collective dynamics depend on the properties of the diffusion function?

More precisely, let $\rho_k(t)$ denote the relative density of active agents at time t with connectivity k. Consequently, $\rho(t) = \sum_k P(k)\rho_k(t)$ is the relative density of active agents at time t. Denote by $\theta(t)$ the probability that any given link points to an active agent at time t. Therefore, the probability that a susceptible agent with k neighbors has exactly a active neighbors at time t is $\binom{k}{a}\,\theta(t)^a(1-\theta(t))^{(k-a)}$, given that this random event follows a binomial distribution with parameters k and $\theta(t)$.

Consider a susceptible agent with k neighbors and a active among them at time t. She becomes active at a rate $\nu f(k,a)$ as specified in equation (1). Hence, the transition rate from susceptible to active, for an agent with connectivity k, is given by

$$\widetilde{g}_{\nu,k}(\theta(t)) \equiv \sum_{a=0}^{k} \nu f(k,a)\binom{k}{a}\,\theta(t)^a(1-\theta(t))^{(k-a)}.$$

The dynamic mean-field equation can thus be written as,

$$\frac{d\rho_k(t)}{dt} = -\rho_k(t)\delta + (1-\rho_k(t))\widetilde{g}_{\nu,k}(\theta(t)). \tag{2}$$

Equation (2) says the following: the variation of the relative density of active agents with k links at time t equals the proportion of susceptible agents with k neighbors at time t who become active (i.e., $(1-\rho_k(t))\widetilde{g}_{\nu,k}(\theta(t))$) minus the proportion of active agents with k neighbors at time t who become susceptible (i.e., $\rho_k(t)\delta$).

The stationary condition is equivalent to $\frac{d\rho_k(t)}{dt} = 0$ in equation (2), for all $k \geq 1$. Therefore, the stationary state must satisfy that

$$\rho_k = \frac{g_{\lambda,k}(\theta)}{1+g_{\lambda,k}(\theta)}, \tag{3}$$

where

$$g_{\lambda,k}(\theta) = \frac{1}{\delta}\widetilde{g}_{\nu,k}(\theta) = \sum_{a=0}^{k} \lambda f(k,a)\binom{k}{a}\,\theta^a(1-\theta)^{(k-a)}.$$

Let $\langle k\rangle$ denote the average connectivity of the network, i.e., $\langle k\rangle = \sum_k kP(k)$. The probability that a node links to another node with connectivity k equals $\frac{kP(k)}{\langle k\rangle}$. Thus, the value of θ can be computed as

$$\theta = \frac{1}{\langle k\rangle}\sum_{k\geq 1} kP(k)\rho_k. \tag{4}$$

Equations (3) and (4) determine the stationary values for θ and $(\rho_k)_{k\geq 1}$. Upon substituting equation (3) into equation (4) we obtain

$$\theta = H_\lambda(\theta), \tag{5}$$

where

$$H_\lambda(\theta) \equiv \frac{1}{\langle k \rangle} \sum_{k \geq 1} k P(k) \frac{g_{\lambda,k}(\theta)}{1 + g_{\lambda,k}(\theta)}. \tag{6}$$

The solutions of equation (5) are the stationary values of θ. Upon replacing θ into equation (3) we can also determine the stationary values $(\rho_k)_{k \geq 1}$ and, by the same token, ρ. Therefore, given a specific connectivity distribution $P(k)$, and a diffusion mechanism m, equation (6) allows us to calculate the value (or values) of ρ in equilibrium. In addition, we can derive general conclusions about the conditions that guarantee the spreading and persistence of the product in the population. We do that next.

3 The Results

In this section we present the main results of the paper. We study how the connectivity distribution and the diffusion mechanism affect the mean-field equilibrium outcomes.

Notice that, in the mean-field dynamics described above, the concept of stationary states only refers to stationary values of θ^* and ρ^* and not to the identities of the individuals choosing each action.

We define next two key concepts in our analysis.

*We say that λ_c is a **critical threshold** if for all $\lambda > \lambda_c$ there exists a stationary state of the dynamics with a positive fraction of active agents, whereas for all $\lambda \leq \lambda_c$ such a stationary state does not exist.*

*We say that λ_d is a **diffusion threshold** if, starting from an infinitesimally small fraction of active agents, the dynamics converges to a state with a positive fraction of active agents for all $\lambda > \lambda_d$, whereas the dynamics converges to a state with zero active agents for all $\lambda \leq \lambda_d$.*

By definition, the diffusion threshold must be at least as high as the critical threshold (i.e., $\lambda_d \geq \lambda_c$).[4] It is worth noting that, for $\lambda \geq 0$ fixed, $0 \leq H_\lambda(\theta) < 1$ for all $\theta \in [0, 1]$. Moreover, it is straightforward to show that $H_{\lambda'}(\theta) \geq H_\lambda(\theta)$, for all $\theta \in [0, 1]$, if and only if $\lambda' \geq \lambda$. This guarantees the existence of a critical threshold (although this threshold might be $+\infty$).

Let us now assume that

$$f(k, 0) = 0 \text{ for all } k \geq 1, \tag{A-1}$$

i.e., having at least one active neighbor is necessary to become active. This implies that the

[4] Note that the critical threshold refers to the existence of a positive stationary state, whereas the diffusion threshold refers to the stability of such a state.

state where all agents are choosing action 0 is a stationary state of the mean-field dynamics.[5] The next result provides the value for the diffusion threshold given any diffusion function satisfying (A-1).[6] In order to obtain the critical threshold more structure on the diffusion function is needed. For this purpose, let us introduce an additional assumption:

For all $k \geq 1$, let $f(k,a)$ be a (weakly) concave function of a. More precisely,

$$f(k,a) - f(k,a-1) \geq f(k,a+1) - f(k,a) \text{ for all } 0 < a < k. \tag{A-2}$$

The interpretation for (A-2) is that, for any given agent, adding one more active neighbor has an impact over her probability of obtaining the product, which is (weakly) decreasing with respect to the existing number of active neighbors.

We have the following result.

Proposition 1 *Let $\lambda^* = \frac{\langle k \rangle}{\sum_k k^2 P(k) f(k,1)}$. Then, the following statements hold:*

(1) If $f(k,a)$ satisfies (A-1) the diffusion threshold is $\lambda_d = \lambda^*$.

(2) If $f(k,a)$ satisfies (A-1) and (A-2) the critical threshold is $\lambda_c = \lambda^*$.

The proof of Proposition 1 is presented in the appendix. For the proof of statement (1) we simply have to find conditions to guarantee that the state where all agents are susceptible is unstable. That is, a finite number of active agents can spread the behavior to an infinite fraction of the population. Regarding statement (2), we show in the appendix that $H_\lambda(\theta)$ is a non-decreasing and concave function. Therefore, as illustrated in Fig. 1, equation (5) has a positive solution if and only if $\frac{dH_\lambda(\theta)}{d\theta}\rfloor_{\theta=0} > 1$ which then allows us to obtain the critical threshold by just solving equation $\frac{dH_{\lambda_c}(\theta)}{d\theta}\rfloor_{\theta=0} = 1$.

Insert Figure 1 about here

Notice that the diffusion threshold does not depend on $f(k,a)$ for $a \neq 1$. Other properties of the diffusion process, however, might depend on these values. Indeed, regarding the critical threshold, notice that if (A-2) holds it coincides with the diffusion threshold. Furthermore, (A-2) also guarantees that there exists a unique positive stationary state of the dynamics for any value of $\lambda > \lambda^*$.[7] We explore next the case in which (A-2) does not hold.

[5] Notice that in any finite system, the process must eventually converge, with probability 1, to an absorbing state where no node is active. Such an outcome, however, depends on a coincidence of events that is extremely improbable if the population is very large. This is why this possibility is implicitly ignored by our mean-field approach.

[6] If, alternatively, (A-1) does not hold (i.e., $f(k,0) > 0$ for some $k \geq 1$) the state $\theta = 0$ is unstable since $H_\lambda(0) > 0$ and therefore our analysis becomes trivial as $\lambda_c = \lambda_d = 0$.

[7] In fact, if (A-1) does not hold, (A-2) implies that there exists a unique (positive) stationary state of the dynamics for all $\lambda > \lambda^* = 0$.

3.1 Non-Concave Diffusion Functions: Hysteresis

In this section we propose an example of a non-concave diffusion function for which the diffusion and critical thresholds do not coincide. Let

$$f(k,a) = \left(\frac{a}{k}\right)^2. \tag{7}$$

Note that, for all $k \geq 1$, $f(k,a)$ is a convex function of a. Hence, for any given agent, adding one more active neighbor has an impact over her probability of obtaining the product, which is increasing with respect to the existing number of active neighbors.

From Proposition 1 and (7), we obtain that diffusion from a small initial seed occurs if and only if

$$\lambda > \lambda_d = \langle k \rangle.$$

To study more general properties of the dynamics one needs to analyze the shape of the family of functions $\{H_\lambda(\theta)\}_{\lambda \geq 0}$. In this case,

$$g_{\lambda,k}(\theta) = \lambda \sum_{a=0}^{k} (\frac{a}{k})^2 \binom{k}{a} \theta^a (1-\theta)^{(k-a)} = \frac{\lambda}{k^2} E[\chi^2],$$

where χ is a random variable that follows a binomial distribution with parameters k and θ. Therefore, the following holds,

$$E[\chi^2] = Var[\chi] + E[\chi]^2 = k\theta(1-\theta) + (\theta k)^2 = (k^2 - k)\theta^2 + k\theta,$$

and thus

$$g_{\lambda,k}(\theta) = \frac{\lambda}{k^2}((k^2-k)\theta^2 + k\theta) = \frac{\lambda}{k}((k-1)\theta^2 + \theta).$$

The function $H_\lambda(\theta)$ can be then written as

$$H_\lambda(\theta) \equiv \frac{1}{\langle k \rangle} \sum_{k \geq 1} kP(k) \frac{\frac{\lambda}{k}((k-1)\theta^2 + \theta)}{1 + \frac{\lambda}{k}((k-1)\theta^2 + \theta)}.$$

The shape of $H_\lambda(\theta)$ depends crucially on $P(k)$. Therefore, for the sake of simplicity, we will focus on two specific types of networks: (i) a scale-free network with $\gamma = 3$, i.e., $P(k) \propto k^{-3}$ and (ii) a homogeneous network.

By means of simple numerical computations, it is straightforward to show that, if $\langle k \rangle$ is sufficiently high (actually, higher than 5) the family of functions $\{H_\lambda(\theta)\}_{\lambda \geq 0}$ exhibits the following pattern for cases (i) and (ii): For low values of λ, $H_\lambda(\theta)$ is convex. As λ increases, $H_\lambda(\theta)$ becomes an *S-shaped* function, i.e., convex for low values of θ and concave for high values of θ. Finally, if λ is sufficiently high, $H_\lambda(\theta)$ is concave. For simplicity, a family of functions $\{H_\lambda(\theta)\}_{\lambda \geq 0}$ satisfying this property is referred as an *S-shaped family.*

Note that, since $\{H_\lambda(\theta)\}_{\lambda \geq 0}$ is an S-shaped family of functions, as illustrated in Fig. 2, there is convergence to a unique stationary state for $\lambda > \lambda_d$. Moreover, there exists a threshold $\widetilde{\lambda}$ (*concavity threshold*) such that, $H_\lambda(\theta)$ is concave for all $\theta \in [0,1]$ if and only if $\lambda > \widetilde{\lambda}$. This value can be derived from equation $H^{''}_{\widetilde{\lambda}}(0) = 0$.

It is straightforward to show

$$H^{''}_{\lambda}(0) = \frac{\lambda}{\langle k \rangle} \sum_{k \geq 1} P(k) \frac{2k(k-1) - 2\lambda k^2}{k},$$

and that the two types of networks considered above satisfy

$$H^{''}_{\lambda_d}(0) > 0,$$

which therefore implies that $\widetilde{\lambda} > \lambda_d$. Hence, for values of λ slightly smaller than λ_d, the functions $H_\lambda(\theta)$ have an S-shape (see, for instance, $H_{\lambda_d -}(\theta)$ in Fig. 2) and thus there are stationary states of the dynamics with a positive fraction of active agents. To reach these states, however, one needs a significantly large fraction of initial active agents. Hence, the critical threshold (although not explicitly determined) is such that $\lambda_c < \lambda_d$.

Insert Figure 2 about here

As a consequence of the previous discussion, and as illustrated in Fig. 4, there is an abrupt increase in the fraction of active agents in the long run of the dynamics $\rho(\lambda)$ (second order phase transition) when λ surpasses λ_d, which contrasts with the transition (first order transition) that occurs when the diffusion function is concave (see Fig. 3), where in both figures we are assuming that, for each value of λ, only an infinitesimal small fraction of individuals are initially active agents (i.e., $\rho_0 \sim 0$). Furthermore, if we consider that there is a large initial fraction of active agents we find that there is a positive fraction of active agents in the long-run of the dynamics, provided $\lambda > \lambda_c$ (see the dashed curve in Fig. 4). The existence of two distinct thresholds (λ_c and λ_d), is a well-known occurrence in many other natural phenomena usually referred as *hysteresis*.

Insert Figures 3 and 4 about here

3.2 Comparative Statics

In this section we analyze how the network structure affects the equilibrium outcomes. We concentrate on the case in which condition (A-1) holds and therefore analyze how the diffusion threshold λ_d changes with the connectivity distribution $P(k)$. It is worth mentioning that, whenever the diffusion function is concave, the results obtained next apply to the concept of the critical threshold as well (as $\lambda_d = \lambda_c$ in this case).

Let us consider two networks with connectivity distributions $P(k)$ and $P'(k)$ respectively, where $P'(k)$ is a Mean Preserving Spread (MPS) of $P(k)$. In other words, these two networks have the same average connectivity but $P(k)$ Second Order Stochastic Dominates (SOSD)

$P'(k)$. This implies, for instance, that $P'(k)$ has a higher variance than $P(k)$.[8] Denote by $\lambda_d(P)$ and $\lambda_d(P')$ the diffusion thresholds obtained for the distributions of P and P' respectively. We have the following.

Proposition 2 *Consider two networks with connectivity distributions $P'(k)$ and $P(k)$, where $P'(k)$ is a MPS of $P(k)$. Then, the following holds:*

(i) if $k^2 f(k,1)$ is convex for all $k \geq 1$ then $\lambda_d(P') \leq \lambda_d(P)$
(ii) if $k^2 f(k,1)$ is concave for all $k \geq 1$ then $\lambda_d(P') \geq \lambda_d(P)$
(iii) if $k^2 f(k,1)$ is linear for all $k \geq 1$ then $\lambda_d(P') = \lambda_d(P)$.

The proof of Proposition 2 is relegated to the appendix. This result indicates that the variance of the connectivity distribution has an effect on the diffusion threshold which depends crucially on the diffusion function. The next corollary is a straightforward consequence of Proposition 2 and stresses this point.

Corollary 3 *Consider two networks with connectivity distributions $P'(k)$ and $P(k)$, where $P'(k)$ is a MPS of $P(k)$. Let $f(k,a)$ be such that $f(k,a) = f(k',a) \equiv f(a)$ $\forall$ $a \geq 0$ and k, $k' \geq a$. Then $\lambda_d(P') \leq \lambda_d(P)$.*

Corollary 3 focuses on diffusion processes where only the absolute number of active neighbors matter. It says, in particular, that the higher the variance of the connectivity distribution the lower the diffusion threshold. To further illustrate this result consider the following networks. Let $m \geq 1$, and consider a scale-free network with connectivity distribution

$$P_{SF}(k) \propto k^{-2.5} \text{ for } k \geq m$$

and $P_{SF}(k) = 0$ otherwise. Consider an exponential network with connectivity distribution

$$P_E(k) \propto e^{-k/2m} \text{ for } k \geq m$$

and $P_E(k) = 0$ otherwise. Finally, consider a homogeneous network such that

$$P_H(k) = 1 \text{ if } k = 3m.$$

These three networks have an average connectivity equal to $3m$. Moreover, $P_{SF}(k)$ is a MPS of $P_E(k)$ and $P_E(k)$ is a MPS of $P_H(k)$.[9] As a consequence of Corollary 3, the diffusion thresholds are ranked as follows:

$$\lambda^*(SF) \leq \lambda^*(P) \leq \lambda^*(H).$$

[8] Jackson and Rogers (2007a) and Jackson and Yariv (2007) also show how one can deduce differences in diffusion properties using stochastic dominance arguments.

[9] To calculate the average connectivity of the connectivity distribution $P(k)$, we approximate $P(k)$ by a continuous distribution where $k \in [m, +\infty)$. First, we compute the multiplicative constant. For example, in the exponential network case, $P(k) = Ce^{-k/2m}$, where C is a normalizing constant. We can then solve for

Note that scale-free networks have infinite variance which implies that $\lambda^*(SF) = 0$.[10] This result extends the classical result in epidemiology; the absence of epidemic threshold for scale-free networks. The intuition of why scale-free networks are extremely vulnerable to diffusion is the following. Due to their high variance, scale-free networks have a significant proportion of hubs, i.e., nodes with very high connectivity compared to the average. For the family of diffusion functions considered in Corollary 3, these nodes turn out to be crucial for the spreading of the product in the population for two reasons. First, they will eventually become active, given their high probability of exposure to other active agents. Second, once they become active, they are capable of influencing many other individuals in the population. This explains why the diffusion threshold is zero for scale-free networks and why, in general, broader connectivity distributions favor the diffusion of the product in the population.

In many social phenomena, relative considerations tend to be important in understanding whether some new behavior is adopted (mainly if there is some persuasion or coordination involved). In other words, not only the total number of active neighbors, but also how this number compares with the number of non-actives, matters. For instance, consider a simple imitation mechanism in which individuals choose one neighbor at random every period, and decide to adopt only if this neighbor adopts. In such a case, the diffusion function equals

$$f(k, a) = \frac{a}{k}, \tag{8}$$

which therefore depends on the fraction of active agents instead of the absolute number.

More generally, one could consider a family of diffusion functions given by

$$f_\beta(k, a) = \frac{a}{k^\beta}, \tag{9}$$

where $\beta \geq 0$ represents the neighborhood effects.

Since $f_\beta(k, a)$ satisfies (A-1) and (A-2) the diffusion and critical thresholds coincide for this case. Note that, if $\beta = 0$, this corresponds to the standard SIS model.[11] One can

C in the equation $1 = \int_m^{+\infty} Ce^{-k/2m} dk$. Notice that, once we know C, we can easily compute the average connectivity as $\int_m^{+\infty} kP(k)dk$. Furthermore, to show that a distribution $P'(k)$ is a MPS of $P(k)$ we simply apply the condition that for all $x \geq 0$ $\int_0^x \overline{P'}(k)dk - \int_0^x \overline{P}(k)dk \geq 0$, where $\overline{P'}(k)$ and $\overline{P}(k)$ are the cumulative distribution functions of $P'(k)$ and $P(k)$ respectively (see e.g., Mas-Colell et al. 1995).

[10] Again, to compute the variance of a scale-free connectivity distribution $P(k) = Ck^{-\gamma}$, we first compute the multiplicative constant. Notice that, once we know C, we can easily compute the average connectivity as $\int_1^{+\infty} kP(k)dk$ and the variance as $\int_1^{+\infty} k^2 P(k)dk - \left(\int_1^{+\infty} kP(k)dk\right)^2$ which converges to infinity for values of $\gamma \in [2, 3]$.

[11] The SIS model assumes that each node is infected from one of his infected neighbors at a rate $\nu > 0$. In

interpret this as a situation where the intensity of each interaction is independent of the total number of neighbors. On the other hand, $\beta = 1$ corresponds to the case in which the intensity of each interaction is inversely proportional to the size of the neighborhood (see (8) and the imitation mechanism proposed above for an alternative interpretation). Moreover, $\beta \in (0,1)$ accounts for intermediate situations (for example, the case in which an individual with twice as many neighbors as another individual needs more active neighbors to obtain the same adoption rate, but not twice as many). Finally, if $\beta > 1$ the rate of adoption decreases with k more than if $\beta \in (0,1]$.

A straightforward comparison among the different diffusion mechanisms specified by (9) is that, the higher the value of β the higher the diffusion threshold.

The next corollary, whose proof is straightforward, investigates further the relationship between the diffusion threshold and the connectivity distribution of the network for this specific class of diffusion functions.

Corollary 4 *Consider two networks with connectivity distributions $P'(k)$ and $P(k)$, where $P'(k)$ is a MPS of $P(k)$. Consider the diffusion function $f(k,a) = \frac{a}{k^\beta}$ where $\beta \geq 0$. Then the following statements holds:*

(i) if $\beta \in [0,1) \cup (2,+\infty)$ then $\lambda_d(P') \leq \lambda_d(P)$

(ii) if $\beta \in (1,2)$ then $\lambda_d(P') \geq \lambda_d(P)$

(iii) if $\beta = 1$ or $\beta = 2$ then $\lambda_d(P') = \lambda_d(P)$.

It is striking to see that β has a non-monotonic effect over the ordering of $\lambda_d(P')$ and $\lambda_d(P)$. For small and large values of β we obtain that broader connectivity distributions have lower critical thresholds (i.e., $\lambda_d(P') \leq \lambda_d(P)$), whereas for intermediate values of β we find that the opposite is true (i.e., $\lambda_d(P') \geq \lambda_d(P)$). Finally, if $\beta = 1$ then $\lambda_d(P') = \lambda_d(P) = 1$, whereas if $\beta = 2$ then $\lambda_d(P') = \lambda_d(P) = \langle k \rangle$.

Let us focus on the case in which $\beta = 1$. In this case,

$$\lambda_d(SF) = \lambda_d(E) = \lambda_d(H) = 1. \tag{10}$$

Therefore, scale-free networks have no comparative advantage for diffusion purposes with respect to other networks. The intuition for such a result hinges upon the existence of two opposite forces. On the one hand, when the connectivity distribution is broader, high-connectivity nodes tend to enhance diffusion (the same way they did when neighborhood effects were absent). However, if neighborhood effects exist, high-connectivity nodes require more neighbors to adopt, which offsets their relevance for the diffusion process. Indeed, as shown in (10) these two forces precisely balance when $\beta = 1$.

a discrete version of the model, this means that if a neighbors are active the probability of becoming active is $1-(1-\nu dt)^a$. In the continuous version, the transition rate $F(\nu,k,a) = \lim_{dt \to 0} \frac{1-(1-\nu dt)^a}{dt} = \nu a$ which corresponds to the linear case $f(a) = a$ in our model.

One possible interpretation for the phenomenon that high-variance networks have lower diffusion thresholds when β is sufficiently high (i.e., $\beta \in (2, +\infty)$) is that low-connectivity nodes become crucial for diffusion in this case, given that relative to other nodes they have a much higher chance of becoming active. Therefore, since broad connectivity distributions have a high number of low-connectivity nodes they enhance diffusion.

Notice that, one can easily compute the diffusion threshold for the diffusion functions given by (9) as

$$\lambda_d = \frac{\langle k \rangle}{\langle k^{2-\beta} \rangle},$$

where

$$\langle k^{2-\beta} \rangle = \sum_{k \geq 1} k^{2-\beta} P(k).$$

In particular, for scale-free networks (i.e., $P(k) \propto k^{-\gamma}$ where $\gamma \in [2, 3]$) we find that $\lambda_d(SF) = 0$ if and only if

$$\beta \leq 3 - \gamma.$$

In other words, the absence of epidemic threshold for scale-free networks only holds when β is below $\beta^* = 3 - \gamma$.

To conclude, it is easy to find examples where the diffusion threshold is neither increasingly nor decreasingly ordered according to MPS. For instance, let $q \in [0, 1]$ and the diffusion function be

$$f_q(k, a) = \begin{cases} 1 \text{ if } \frac{a}{k} > q \\ 0 \text{ if } \frac{a}{k} \leq q \end{cases}.$$

Consider the following simple family of connectivity distributions. Given $r \in [0, \langle k \rangle]$, the connectivity distribution is

$$P_a(k) = \begin{cases} \frac{1}{2} \text{ if } k = \langle k \rangle + r \\ \frac{1}{2} \text{ if } k = \langle k \rangle - r \end{cases}$$

These distributions have all the same average connectivity. Note that the connectivity of individuals can take only two values which are equidistant from the average $\langle k \rangle$; half of the population has connectivity $\langle k \rangle + r$, whereas the other half has connectivity $\langle k \rangle - r$. Thus, the larger r the broader the connectivity distribution. In such a case, the diffusion threshold is given by:

$$\lambda_d = \frac{\langle k \rangle}{\sum\limits_{k \geq 1}^{[1/q]} k^2 P(k)}$$

where $[1/q]$ stands for the highest integer weakly below $1/q$. For instance, if $\langle k \rangle = 10$ and $q = 1/8$ then the lowest diffusion threshold is attained when $r = 2$, which corresponds to a distribution with intermediate variance. Notice that if $r = 0$ or $r = 1$ then $\lambda_d = +\infty$, whereas if $r \geq 2$ then $\lambda_d = \frac{2\langle k \rangle}{(\langle k \rangle - r)^2}$ is increasing with r.

4 Concluding Remarks

In this paper we present a wide range of diffusion processes and study their performances on different types of networks. We concentrate on the analysis of networks characterized by their connectivity distributions and apply mean-field approximations to obtain analytical results. We find the threshold for the spreading rate of the new behavior above which it can spread and become persistent in the population. We show that the concavity of the diffusion function suffices to guarantee the existence of a unique long-run prediction of the dynamics. We also provide an illustrative example of a non-concave diffusion function where initial conditions matter and therefore hysteresis occurs. Furthermore, this paper sheds some light on the effect that the heterogeneity of agents in terms of their connectivity has on collective outcomes and how this depends on the properties of the diffusion mechanism.

The theoretical results are derived using the so-called mean-field theory. This methodology is useful to address questions that otherwise would be intractable. The mean-field simplification of the model is a standard tool from statistical physics which provides a reasonable guide of the qualitative behavior of complex dynamics (see e.g., Goldenfeld, 1992). In the context of diffusion on networks, one conjectures that the mean-field model approximates well the dynamics on fixed networks but generated randomly and characterized by a given connectivity distribution (random networks). The reason for this is that random networks lack correlation structure, that is, they have no recurrent or easily discernible pattern which makes a mean-field assumption more appropriate. Moreover, while having significant node heterogeneity, they exhibit many large-scale regularities which can be studied through statistical analysis. Nevertheless, the theoretical comparison between most of the results obtained through a mean-field model and a model where the network is a random network remains still as an open question. One can proceed by comparing the mean-field outcomes with those obtained through simulations on random networks. Following this line of research, some simulations can be found in López-Pintado (2004, 2007) and Pastor-Satorrás and Vespignani (2001), among others.

To conclude, it is worth commenting on other works dealing with mean-field approximations of a network model. For instance, Barabási and Albert (1999), Jackson and Rogers (2007b) and Newman et al. (2000) propose models of network formation where individuals form new links according to a certain stochastic process. In order to derive some analytical results, mean-field theory is used. Jackson and Yariv (2007) propose a diffusion model, similar to the one studied here but, unlike in our model, the diffusion mechanism is such that the probability of choosing an action in a given period does not depend on whether the agent is currently choosing action 0 or 1. As Jackson and Yariv (2007) explain, in their framework, the equilibrium outcomes of the mean-field dynamics could also be thought of as the symmetric Bayesian equilibrium outcomes of a Bayesian game where agents, characterized by their connectivities (types), simultaneously choose their actions knowing the connectivity distribution in the population, and assuming that connectivities are independently allocated throughout the network (see also Galeotti et al., 2005 for another use of this same approach). Finally, Dodds and Watts (2004, 2005) analyze by means of a mean-field approach and relying on simulations, a generalized model of contagion that extends the SIS framework by incorporating individual memory of exposure to a contagious entity.

5 Appendix

Proof of Proposition 1:

To prove part (1) assume, for simplicity, that only one agent is active in the initial period. Consider one of the agents to whom this initial seed is linked to. With probability $\frac{kP(k)}{\langle k \rangle}$ this other agent has connectivity k and, given that only one of her neighbors is active, she will become active at a rate $\nu f(k,1)$. Moreover, while she is active (i.e., in an interval of time $\frac{1}{\delta}$) she contacts other individuals at a rate k. Overall, one can assume that the expected number of new *active links* generated by one initial active link is $\sum_{k \geq 1} \frac{kP(k)}{\langle k \rangle} \nu f(k,1) k \frac{1}{\delta}$, where an active link is defined as a link where one of the individuals involved in it is active. Then, the spreading of the product in the population is guaranteed if and only if the expected number exceeds 1. That is,

$$\sum_{k \geq 1} \frac{kP(k)}{\langle k \rangle} \nu f(k,1) k \frac{1}{\delta} > 1,$$

or, equivalently,

$$\lambda > \lambda^* = \frac{\langle k \rangle}{\sum_{k \geq 1} k^2 P(k) f(k,1)}. \tag{11}$$

To show that this condition is necessary, let k_0 be the connectivity of the initial seed. In other words, there are initially k_0 active links. If condition (11) does not hold then the expected number of active links decreases with time. In a discrete version of the dynamics this implies that the number of active links in the next period is k_1, such that $k_0 > k_1$. We can apply the same argument again and show that the expected number of active links in period 2 satisfies $k_1 > k_2$, and so on. Thus, eventually $k_t = 0$ and therefore the dynamics converges to a state where $\theta^* = 0$. If, on the other hand, condition (11) holds then the expected number of active links increases with time. That is, $k_0 < k_1$ and, more generally, $\{k_t\}_{t \geq 0}$ is an increasing sequence of natural values which converges to infinity. This implies that in the long run of the dynamics there is a positive fraction of active links and therefore $\theta^* = 0$ is unstable.

To prove part (2) notice that (A-1) implies that $\theta = 0$ is a stationary state of the dynamics (i.e., $H_\lambda(0) = 0$). In addition, $0 \leq H_\lambda(\theta) < 1$ for all $\theta \in [0,1]$ since $g_{\lambda,k}(\theta) \geq 0$. It is straightforward to show that, for every given $\theta \in [0,1]$, $H_\lambda(\theta)$ is a non-decreasing function of λ. Moreover, let us prove that given $\lambda \geq 0$ and (A-2), $H_\lambda(\theta)$ is a non-decreasing and (weakly) concave function of θ.

Recall that,

$$H_\lambda(\theta) \equiv \frac{1}{\langle k \rangle} \sum_{k\geq 1} kP(k) \frac{g_{\lambda,k}(\theta)}{1+g_{\lambda,k}(\theta)}, \tag{12}$$

where

$$g_{\lambda,k}(\theta) = \sum_{a=0}^{k} \lambda f(k,a) \binom{k}{a} \theta^a (1-\theta)^{(k-a)}.$$

Then, using equation (12), we know that

$$H'_\lambda(\theta) = \frac{1}{\langle k \rangle} \sum_{k\geq 1} kP(k) \frac{g'_{\lambda,k}(\theta)}{(1+g_{\lambda,k}(\theta))^2}, \tag{13}$$

where

$$\begin{aligned} g'_{\lambda,k}(\theta) &= \sum_{a=0}^{k} \lambda f(k,a) \binom{k}{a} \left(a\theta^{a-1}(1-\theta)^{(k-a)} - \theta^a (k-a)(1-\theta)^{(k-a-1)}\right) \\ &= \sum_{a=0}^{k-1} (\lambda(a+1) f(k,a+1) \binom{k}{a+1} - \lambda(k-a) f(k,a) \binom{k}{a}) \theta^a (1-\theta)^{(k-a-1)} \end{aligned} \tag{14}$$

and since

$$(a+1)\binom{k}{a+1} = (k-a)\binom{k}{a} = \frac{k!}{a!(k-a-1)!},$$

then

$$g'_{\lambda,k}(\theta) = \sum_{a=0}^{k-1} \frac{k!}{a!(k-a-1)!} \lambda(f(k,a+1) - f(k,a)) \theta^a (1-\theta)^{(k-a-1)}$$

which is non-negative and therefore implies that $H_\lambda(\theta)$ is non-decreasing.

Moreover,

$$H''_\lambda(\theta) = \frac{1}{\langle k \rangle} \sum_{k\geq 1} kP(k) \frac{g''_{\lambda,k}(\theta)(1+g_{\lambda,k}(\theta)) - 2(g'_{\lambda,k}(\theta))^2}{(1+g_{\lambda,k}(\theta))^3},$$

where

$$\begin{aligned} g''_{\lambda,k}(\theta) &= \sum_{a=0}^{k-1} \frac{k!}{a!(k-a-1)!} \lambda(f(k,a+1) - f(k,a)) \\ &\quad (a\theta^{a-1}(1-\theta)^{(k-a-1)} - \theta^a (k-a-1)(1-\theta)^{(k-a-2)}) \end{aligned}$$

or equivalently

$$\begin{aligned} g''_{\lambda,k}(\theta) &= \sum_{a=0}^{k-2} \lambda \frac{k!(a+1)}{(a+1)!(k-a-2)!}(f(k,a+2)-f(k,a+1))\theta^a(1-\theta)^{(k-a-2)} \\ &\quad -\lambda \frac{k!(k-a-1)}{a!(k-a-1)!}(f(k,a+1)-f(k,a))\theta^a(1-\theta)^{(k-a-2)} \\ &= \sum_{a=0}^{k-2} \lambda((f(k,a+2)-f(k,a+1))-(f(k,a+1)-f(k,a))) \\ &\quad \frac{k!}{a!(k-a-2)!}\theta^a(1-\theta)^{(k-a-2)}. \end{aligned}$$

Observe that assumption (A-2) implies that $g''_{\lambda,k}(\theta) \leq 0$ which in particular implies that $H_\lambda(\theta)$ is concave. Finally, notice that, if $H_\lambda(\theta)$ is non-decreasing and concave then there exists a non-null stationary state of the dynamics if and only if

$$H'_\lambda(0) > 1,$$

or, analogously,

$$H'_\lambda(0) = \frac{\lambda \sum_{k\geq 1} k^2 P(k) f(k,1)}{\langle k \rangle} > 1 \Leftrightarrow \lambda > \frac{\langle k \rangle}{\sum_{k\geq 1} k^2 P(k) f(k,1)}$$

which completes the proof. □

Proof of Proposition 2:

The diffusion threshold equals

$$\lambda_d = \frac{\langle k \rangle}{\sum_{k\geq 1} k^2 P(k) f(k,1)}.$$

Then, using a well-known characterization of the concept of Mean Preserving Spread (see e.g., Mas-Colell et al., 1995), we know that for any concave function $u(k)$

$$\sum_{k\geq 1} u(k)P(k) \geq \sum_{k\geq 1} u(k)P'(k)$$

given that $P'(k)$ is a MPS of $P(k)$. Therefore, if $k^2 f(k,1)$ is concave then $\lambda_d(P) \leq \lambda_d(P')$. If $k^2 f(k,1)$ is convex (since this implies that $-k^2 f(k,1)$ is concave) then $\lambda_d(P) \geq \lambda_d(P')$. Finally, if $k^2 f(k,1)$ is linear then it must be the case that $k^2 f(k,1) = Ak + B$, where A and B are constants. This implies that $\lambda_d(P) = \lambda_d(P') = \frac{\langle k \rangle}{A\langle k \rangle + B}$. Also note that if $f(k,a)$ satisfies (A-2), the same results apply for the critical threshold λ_c.

□

References

[1] Albert, R., Jeong, H., Barabási, A. L., 1999. Diameter of the World Wide Web. Nature 401, 130-131.

[2] Aguirre, B. E., Quarantelli, E. L., Mendoza, J. L., 1988. The collective behavior of fads: the characteristics, effects, and career of streaking. Amer. Econ. Rev. 53, 569-584.

[3] Benaim, M., Weibull, J., 2003. Deterministic approximation of stochastic evolution in games. Econometrica 71, 873-903.

[4] Barabási, A. L., Albert, R., 1999. Emergence of scaling in a random network. Science 286, 509.

[5] Blume, L., 1995. The statistical mechanics of best-response strategy revision. Games Econ. Behav. 11, 111-145.

[6] Dodds, P.S., Watts, D. J., 2004. Universal behavior in a generalized model of contagion. Phys. Rev. Lett. 92, 218701.

[7] Dodds, P.S., Watts, D. J., 2005. A generalized model of social and biological contagion. Journal of Theoretical Biology 232, 587-604.

[8] Ellison, G., 1993. Learning, local interaction, and coordination. Econometrica 61, 1047-1071.

[9] Galeotti, A., Goyal, S., Jackson, M. O., Vega-Redondo, F., Yariv, L., 2006. Network games. Working Paper, Caltech.

[10] Glaeser, E. L., Sacerdote, B., Scheinkman, J. A., 1996. Crime and social interaction. Quart. J. Econ. 111, 507-48.

[11] Goldenfeld, N., 1992. Lectures on phase transitions and the renormalization group. Addison-Wesley, New York, NY,.

[12] Jackson, M. O., 2006. The economics of social networks. R. Blundell, W. Newey, T. Persson (Eds.), Proceedings of the 9th World Congress of the Econometric Society, Cambridge University Press, forthcoming.

[13] Jackson, M.O., Rogers, B., 2007a. Relating network structure to diffusion properties through stochastic dominance. Advances in Theoretical Economics 7, 1-13.

[14] Jackson, M.O., Rogers, B., 2007b. Meeting strangers and friends of friends: how random are socially generated networks? Amer. Econ. Rev. 97, 890-915.

[15] Jackson, M. O., Yariv, L., 2007. Diffusion of behavior and equilibrium properties in network games. Amer. Econ. Rev. (Papers and Proceedings) 97, 92-98.

[16] Lijeros, F., Edling, C. R., Amaral, L. A. N., Stanley, H. E., Aberg, Y., 2001. The web of human sexual contacts. Nature 411, 907-908.

[17] López-Pintado, D., 2004. Diffusion in complex social networks. IVIE Discussion Paper. WP-AD 2004-33.

[18] López-Pintado, D., 2006. Contagion and coordination in random networks. Int. J. Game Theory 34, 371-381.

[19] López-Pintado, D., 2007. The spread of free-riding behavior in a social network. Working Paper, Universitat Autònoma de Barcelona.

[20] Mas-Colell, A., Winston, M., Green, J., 1995. Microeconomics Theory. Oxford University Press, Oxford.

[21] Morris, S., 2000. Contagion. Rev. Econ. Stud. 67, 57-78.

[22] Newman, M.E.J., Moore, C., Watts, D. J., 2000. Mean-field solution of a small-world network model. Phys. Rev. Lett. 84, 3201-3204.

[23] Pastor-Satorrás, R., Vespignani, A., 2001. Epidemic spreading in scale-free networks. Phys. Rev. Lett. 86, 3200-3203.

[24] Rogers, E. M., 1995. The Diffusion of Innovations. The Free Press, New York, NY.

[25] Vega-Redondo, F., 2006. Complex social networks. The Econometric Society Monograph Series, Cambridge University Press.

[26] Watts, D. J., 2002. A simple model of information cascades on random networks. Proc. Natl. Acad. Sci. 99, 5766-5771.

[27] Young, H.P., 1998. Individual strategy and social structure: an evolutionary theory of institutions. Princeton University Press, Princeton, NJ.

[28] Young, H.P., Burke, M.A., 2001. Competition and custom in economic contracts: a case study of Illinois agriculture. Amer. Econ. Rev. 91, 559-573.

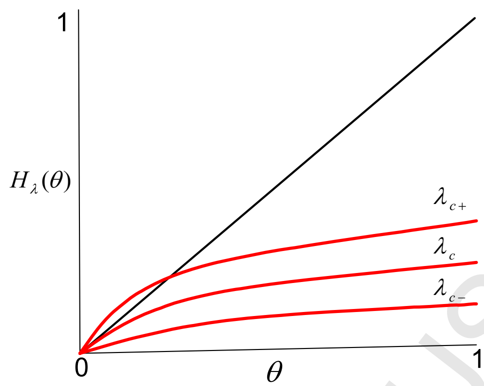


Figure 1: Representation of $H_\lambda(\theta)$ for a diffusion function satisfying (A-1) and (A-2) when (i) λ equals the critical threshold λ_c (ii) λ is slightly above the critical threshold (i.e., $\lambda = \lambda_{c+}$) and (iii) λ is slightly below the critical threshold (i.e., $\lambda = \lambda_{c-}$).

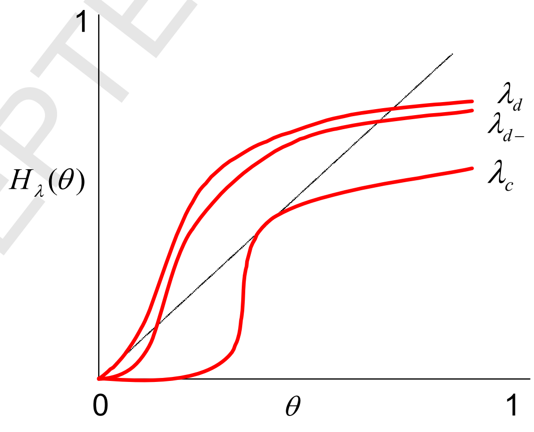


Figure 2: Representation of an S-shaped family of functions $H_\lambda(\theta)$ when the critical and diffusion thresholds do not coincide. We plot three curves corresponding to (i) λ equals the diffusion threshold λ_d (ii) λ equals the critical threshold λ_c and (iii) λ is slightly below the diffusion threshold (denoted by λ_{d-}).

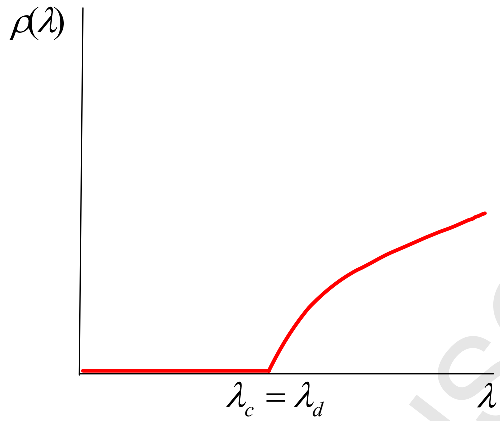


Figure 3: The curve in the graph represents the fraction of active agents in the long run of the dynamics $\rho(\lambda)$ as a function of the spreading rate λ for a diffusion function satisfying assumptions (A-1) and (A-2).

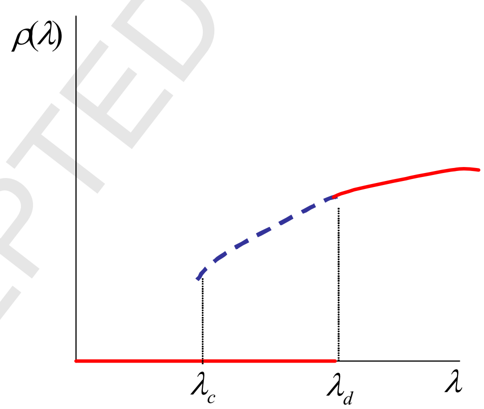


Figure 4: The continuous line in the graph represents the fraction of active agents in the long run of the dynamics as a function of the spreading rate λ under the assumption that initially only an infinitesimal fraction of individuals are active. The dashed line corresponds to the fraction of active agents in the long run of the dynamics as a function of λ when the initial fraction of active agents corresponds with $\rho(\lambda + \varepsilon)$, where ε is sufficiently small. Another interpretation of this graph is that whenever $\lambda_c < \lambda < \lambda_d$ there are two stable states of the dynamics; one where $\rho^* = 0$ and another where $\rho^* > 0$.