Voting Over Intervals*

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November 16, 2010

Abstract

We consider a collection of problems where a group of individuals must choose an interval, e.g., a range of values for a policy, a time frame to deliver a service, a location and size for a facility. We study the existence of strategy-proof voting procedures in that context. Our main characterization result depends crucially on the number of choices that voters are presented with. This allows us to remark that controlling for the number of alternatives can be a determinant tool for a designer. In our context, there is a choice of alternative voting procedures when the range is constrained to be small, while unanimity remains as the only possibility in contexts where agents are a priori given more choices.

JEL classification: D71.
Keywords: Intervals; Voting under Constraints; Voting by Committees; Strategy-proofness; Additive Preferences; Separable Preferences.

1 Introduction

Consider the following social choice problems. (i) A Central Bank has set up a committee to decide on which range to let the value of the currency float without interventions. The policy space is divided in different nonoverlapping intervals,

*I would like to specially thank Salvador Barberà for his helpful guidance. For helpful comments I also thank Vikram Manjunath, Jordi Massó, Pedro Rey, Shigehiro Serizawa, William Thomson and two anonymous referees. In addition, I would like to acknowledge Antonio Romero-Medina for stating a problem that originally motivated the writing of this paper. Of course, the usual disclaimer applies. Finally, I gratefully acknowledge financial support from the Generalitat de Catalunya through grant FI Agaur.

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and the committee members vote over these different ranges. (ii) Based on the votes of individuals over possible intervals, a municipal board must decide on the location and size of a hospital. This is modeled by the number and location of plots to be appropriated, each of which is represented by an interval on the line. (iii) A municipal board must regulate the night clubs' opening and closing hours. All of these problems have the same structure: a group of voters must choose intervals whose union must also be an interval. This paper deals with this kind of problems. In particular, we look for strategy-proof voting procedures that choose intervals[1].

Finding strategy-proof voting procedures is not a trivial problem. Moreover, it is impossible to achieve this aim if any preference domain is considered (see Gibbard (1973) and Satterthwaite (1975)). Therefore, we restrict individuals' preferences in the following way: whenever an individual adds to a set of intervals, an "acceptable" interval, then this individual prefers this new set to the former one. In this way, we rule out interaction effects among intervals. Given this assumption, we know that if a strategy-proof voting procedure exists, then this must be in the class of voting by committees (Barberà et al. (1991)). A voting procedure in this class works as is voters were asked to vote for each of the nonoverlapping intervals sequentially.

However, not any voting procedure in the class of voting by committees always chooses an interval. To get a feeling for the kind of difficulties we will face and also of our suggested ways out, consider the following example. A municipal board, composed by individuals 1, 2 and 3, must regulate the night clubs' opening and closing hours. The night, i.e. the range of time from 11 p.m to 6 a.m., is divided in three intervals: a_1 (11 p.m. to 1 a.m.), a_2 (1 a.m. to 3 a.m.) and a_3 (3 a.m. to 6 a.m.). Individuals' preferences are defined over the family of all the subsets of the set of intervals, where the empty set is interpreted as the situation where the night clubs must remain closed. Consider the preferences presented in table 1. For instance, for individual 1 the most preferred alternative is to allow the night clubs to be open all night (i.e. the set $\{a_1, a_2, a_3\}$), and this is preferred to $\{a_2, a_3\}$, which is preferred to $\{a_1, a_2\}$, and so on. Assume that the community uses the procedure of voting by quota 2, that is, an interval is chosen if at least two individuals vote for it[2]. Notice that, under this procedure the individuals will end up choosing the alternative $\{a_1, a_3\}$ that is not feasible, since it is not an interval. So here we have an example where a well known strategy-proof voting procedure fails to attain a feasible result. In this case, we could have guaranteed a satisfactory outcome by the following procedure: consider a voting procedure that requires quota 2 for the intervals a_1 and a_3, and quota 1 for the interval a_2; then the chosen alternative is $\{a_1, a_2, a_3\}$, which

[1] A voting procedure that is not manipulable, i.e. there is no individual that has incentives to misrepresent her preferences when submitting her vote, is called strategy-proof.

[2] For the sake of the presentation, we are going to provide the formal definition later on. Just for reference, let us point out that voting by quota is a particular kind of voting by committees.

is an interval[3].

In this paper, we examine the structure of domains and ranges that will let us design voting procedures that are strategy-proof and that always choose an interval[4]. We consider the following domains of preferences: additive preferences and separable preferences with intervals in the tops. In order to attain these characterizations we rely on previous results from the literature on voting by committees under constraints (in particular, we rely on Barberà et al. (1997 and 2005); but see e.g.: Barberà et al. (1991 and 1998), Le Breton and Sen (1999), and Nehring and Puppe (2007)). We obtained two types of results: first a negative result and then a positive result. Since additive preferences are a subset of separable preferences, we use additive preferences for negative results and separable preferences for positive results. In this way, our results gain strength.

Table 1.

Individual 1	Individual 2	Individual 3
$\{a_1, a_2, a_3\}$	$\{a_3\}$	$\{a_1\}$
$\{a_2, a_3\}$	$\emptyset$	$\emptyset$
$\{a_1, a_2\}$	$\{a_2, a_3\}$	$\{a_1, a_2\}$
$\{a_1, a_3\}$	$\{a_1, a_3\}$	$\{a_1, a_3\}$
$\{a_2\}$	$\{a_1, a_2, a_3\}$	$\{a_1, a_2, a_3\}$
$\{a_3\}$	$\{a_2\}$	$\{a_2\}$
$\{a_1\}$	$\{a_1\}$	$\{a_3\}$
$\emptyset$	$\{a_1, a_2\}$	$\{a_2, a_3\}$

Our negative result shows that only dictatorial rules are strategy-proof when restricting the domain to additive preferences. On the other hand, the class of strategy-proof voting procedures that ensures an interval as an outcome is richer assuming separable preferences with intervals in the tops. Indeed, from the technical point of view, this paper is mainly a nontrivial application of previous results found in the literature of voting under constraints. However, the context of this paper, allows us to remark that the main characterization result depends crucially on the number of choices that voters are presented with. Specifically, our positive result indicates that a particular class of strategy-proof voting procedures enlarges when there are few alternatives. This is an appealing conclusion, because it draws attention to the design of the voting procedures; in the sense, that the number of alternatives that we put into consideration may affect the number of strategy-proof voting procedures that we could implement. This might be a central issue if the designer would like to have some freedom

[3] Voting by quota 1 means that under this voting procedure an interval is chosen if at least one individual votes for it.

[4] Of course, there may be other significant restrictions when a group of individuals is trying to choose an interval, like imposing that the length of the chosen interval should not exceed a certain value. However, adding other restrictions would almost surely lead us to negative results.

when choosing among different voting procedures.

The paper is organized as follows. In section 2, we introduce the notation and some basic definitions. In section 3, we present a negative and a positive result. In section 4, we characterize the class of voting by quota that always chooses an interval. In section 5, we state some conclusions.

2 Notation and Basic Definitions

Consider a set of n voters N, with n greater than or equal to 2 ($\#N = n \geq 2$). Given an interval $[0, H]$, $H > 0$, let K be a finite partition in k subintervals of $[0, H]$, with $k > 2$.[5] We denote by $a_1, a_2, ..., a_k$ the elements of K, with the subindices respecting the natural order given by the position of each interval in the real line. Generic elements of K will be denoted by a_l, a_m and a_q. Alternatives are subsets of K and are denoted by A, $B, C, D, A', B', C', D', A''$, and so on[6].

In this paper, we are going to pay special attention to those alternatives that are intervals. The following definition identifies this type of alternatives.

Definition 1 *An alternative $A \in 2^K$ is an interval if $A = \emptyset$ or if for all $a_l, a_m \in A$ satisfying $1 \leq l \leq m \leq k$ we have that $a_q \in A$ for all q such that $l \leq q \leq m$.*

We denote by $\mathcal{C}$ the class of all alternatives that are intervals.

Each voter i has a complete, transitive and asymmetric preference relation on 2^K denoted by P_i. Let $\mathcal{P}_i$ be the class of all possible P_i, and a generic $\mathcal{P}_i$ is denoted by $\mathcal{P}$ (likewise, P denotes a generic preference relation). Although, our focus is on voting procedures that choose an interval, there might be voters that still prefer alternatives that are not an interval rather than an interval. That is why we consider this general setting, because we want individuals to be able to rank any possible set of alternatives. Nevertheless, even if there is a group of voters with this characteristic, we only consider voting procedures such that the only information that is taken into account from a voter's preference relation is her most preferred interval. In this section, we argue why this is crucial. On the other hand, in the next section we restrict to preferences that have intervals in their top and we show the importance of this assumption for obtaining positive results.

[5]When $k \leq 2$, any set of subintervals is going to be itself an interval. That's why we discard those trivial cases.

[6]It is worth to mention that, there exists an alternative way of presenting this set of alternatives. We can reformulate the model such that each alternative is a vertex of an hypercube. In this framework, individuals vote over these vertexes, but those vertexes that represent intervals are the ones that receive special attention (for this geometrical reformulation see e.g.: Aswal et al. (2003) and Barberà et al. (1997 and 2005)).

A preference profile P^n is a n-tuple of preference relations, and the class of all possible preference profiles $\mathcal{P}^n$ is defined by $\mathcal{P}^n = \times_{i \in N} \mathcal{P}_i$. A voting procedure is a function $f : \mathcal{P}^n \to 2^K$. Since we are interested in those voting procedures that select an interval, we need to define the range of this class of voting procedures. The range of the voting procedure f is denoted by $\mathcal{R}_f$, and is defined by $\mathcal{R}_f = \{A \in 2^K \mid \text{there exists } P^n \in \mathcal{P}^n \text{ such that } f(P^n) = A\}$.

Notice, that we study voting procedures whose ranges contain only intervals. On top of that, we would like to have a property that ensures that any interval belongs to the range of this type of voting procedures. This property is desirable because, if an interval never arises from a voting procedure, then this can be interpreted as an extra constraint. In this sense, we say that a voting procedure satisfies voter sovereignty on $\bar{\mathcal{C}} \subseteq 2^K$ if, for each $A \in \bar{\mathcal{C}}$, there exists $P^n \in \mathcal{P}^n$ such that $f(P^n) = A$. The following definition presents our main object of study.

Definition 2 *A voting procedure, $f : \mathcal{P}^n \to 2^K$, is satisfactory if it respects voter's sovereignty on $\mathcal{C}$, and for all $A \in \mathcal{R}_f$, we have that A is an interval.*

We can interpret this restriction on a voting procedure as: 2^K representing the class of conceivable alternatives and $\mathcal{C}$ representing the class of the feasible ones. In this sense, we say that a voting procedure respects feasibility if it always chooses an interval. In the rest of this section, we present some standard concepts of the voting by committees literature and also the concept of strategy-proofness.

Informally, a voting procedure is voting by committees if and only if we can define winning coalitions of voters for each interval in K, such that, an interval is chosen if a winning coalition votes for it. We now proceed formally. A committee is a pair $C = (N, \mathcal{W})$, where $\mathcal{W}$ is a nonempty family of nonempty coalitions of N. This $\mathcal{W}$ satisfies coalition monotonicity: if $W \in \mathcal{W}$ and $W \subseteq W'$, then $W' \in \mathcal{W}$. We say that coalitions in $\mathcal{W}$ are winning. $W \in \mathcal{W}$ is a minimal winning coalition if and only if $W' \subsetneq W$ implies $W' \notin \mathcal{W}$ ($W' \subsetneq W$ means $W' \subseteq W$ and $W' \neq W$). Let $\tau_{\bar{\mathcal{C}}}(P)$ denote the maximal element of $\bar{\mathcal{C}} \subseteq 2^K$ according to the preference relation P. We consider a generic $\bar{\mathcal{C}} \subseteq 2^K$, because sometimes we will take $\bar{\mathcal{C}} = 2^K$ and in other parts of the paper we will specify $\bar{\mathcal{C}} = \mathcal{C}$.

Definition 3 *Given $\bar{\mathcal{C}} \subseteq 2^K$, a voting procedure $f : \mathcal{P}^n \to 2^K$ is voting by committees with focus on $\bar{\mathcal{C}}$, if for each $a_l \in K$, there exists a committee $C_{a_l} = (N, \mathcal{W}_{a_l})$ such that: for all profiles $P^n \in \mathcal{P}^n$, $a_l \in f(P^n)$ if and only if $\{i \in N \mid a_l \in \tau_{\bar{\mathcal{C}}}(P_i)\} \in \mathcal{W}_{a_l}$.*

Definition 4 *Given $\bar{\mathcal{C}} \subseteq 2^K$, a voting procedure $f : \mathcal{P}^n \to 2^K$ is dictatorial with focus on $\bar{\mathcal{C}}$, if there exists an $i \in N$ such that: for all profiles $P^n \in \mathcal{P}^n$ and all $a_l \in K$, $a_l \in f(P^n)$ if and only if $a_l \in \tau_{\bar{\mathcal{C}}}(P_i)$.*

Notice one important feature of Definition 3. An element a_l is going to be elected if and only if, for each of the voters of a winning coalition for a_l, a_l belongs to the top of the voter's preference on $\bar{\mathcal{C}}$ ($a_l \in \tau_{\bar{\mathcal{C}}}(P_i)$). In other words, a voting procedure in the class of voting by committees with focus on $\bar{\mathcal{C}}$ does not take into account the preferences' unrestricted tops, instead it considers the preferences' tops restricted to $\bar{\mathcal{C}}$. This is important because if this restriction is not made, even the dictatorial procedure may produce results that are not an interval. On the other hand, a voting procedure that is dictatorial with focus on $\mathcal{C}$ always choose an interval, even if the dictator has preferences with a top that is not an interval.

In order to escape from manipulability, we introduce the notion of strategy-proofness relative to a specific domain of preferences. By the Gibbard-Satterthwaite theorem (Gibbard (1973) and Satterthwaite (1975)), we know that voting by committees is manipulable on $\mathcal{P}$. Since we are interested in those domains on which the voting procedure turns out to be strategy-proof, we introduce a generic subset of $\mathcal{P}$ that we will denote by $\hat{\mathcal{P}}$. $\hat{\mathcal{P}}$ defines the following domain restriction $\hat{\mathcal{P}}^n = \times_{i \in N} \hat{\mathcal{P}}_i$, where $\hat{P}_i, \hat{P}'_i$, etc., are elements of $\hat{\mathcal{P}}_i$. And in order to stress the preference relation of a voter $i \in N$, we denote a particular profile of $\hat{\mathcal{P}}^n$ by $[\hat{P}_{-i}, \hat{P}_i]$; where $\hat{P}_{-i} \in \hat{\mathcal{P}}_{-i}$ and $\hat{\mathcal{P}}_{-i} = \times_{j \in N \setminus \{i\}} \hat{\mathcal{P}}_j$.

Definition 5 *A voting procedure, $f : \mathcal{P}^n \to 2^K$, is manipulable on $\hat{\mathcal{P}}$ at any $\hat{P}^n \in \hat{\mathcal{P}}^n$ by any $i \in N$ via any $\hat{P}'_i \in \hat{\mathcal{P}}_i$ if $f([\hat{P}_{-i}, \hat{P}'_i]) \hat{P}_i f(\hat{P}^n)$. A voting procedure f is strategy-proof on $\hat{\mathcal{P}}$ if it is not manipulable on $\hat{\mathcal{P}}$.*

This definition allows us to consider different domains. We are going to concentrate on domains that guarantees strategy-proofness. In particular, we are going to consider separable preferences and additive preferences.

Definition 6 *A preference relation P on 2^K is separable if for all $A \in 2^K$ and all $a_l \notin A$ ($a_l \in K$), $A \cup \{a_l\} P A$ if and only if $\{a_l\} P \emptyset$.*

The set of separable preferences will be denoted by $\mathcal{P}^S$. Let $\bar{\mathcal{P}}^S$ be the set of separable preferences with intervals in the top, i.e. $\bar{\mathcal{P}}^S = \{P \in \mathcal{P}^S \mid \tau_{2^K}(P) \in \mathcal{C}\}$; and $(\bar{\mathcal{P}}^S)^n = \times_{i \in N} \bar{\mathcal{P}}_i^S$.

Definition 7 *A preference relation P on 2^K is additive if there exists a function $u : K \to \mathbb{R}$ such that for all $A, B \subseteq K$, $A\ P\ B$ if and only if* $\sum_{a_l \in A} u(a_l) > \sum_{a_m \in B} u(a_m)$.

The set of additive preferences will be denoted by $\mathcal{P}^A$, and $(\mathcal{P}^A)^n = \times_{i \in N} \mathcal{P}_i^A$. As we mentioned in the introduction, in order to obtain stronger results we consider additive preferences for the negative ones and separable preferences for the positive ones. In the next section, we present these findings.

3 Two Characterization Results

Let us first summarize the main ideas of the paper. We look for strategy-proof voting procedures that always choose an interval. In order to attain this aim, we consider separable preferences as an interesting domain restriction. Under this assumption, there is a loss of generality, since we rule out the existence of interaction effects between intervals; but we know that under this domain of preferences, strategy-proof voting procedures exist. Moreover, we know that these voting procedures are of the class of voting by committees (Barberà et al. (1991)). Nevertheless, this strategy does not guarantee that the voting procedure always selects an interval. Hence, we consider voting by committees with focus on $\mathcal{C}$ instead of voting by committees with focus on 2^K. As we mentioned earlier, if we do not do that, even a dictatorial voting procedure might not satisfy feasibility. However, in this section we argue that even with this restriction on the voting procedures, the choice made may fail to be an interval. Therefore, we present a class of voting by committees with focus on $\mathcal{C}$ called "coordinated", that "coordinates" the votes of voters, in the sense that we define in this section, in order to ensure feasible results. Assuming preferences with intervals in the tops, our positive result (Theorem 2) shows that the only strategy-proof voting procedures that are satisfactory are the "coordinated" ones. On top of that, Proposition 3 shows that, for a class of "coordinated" voting procedure, there is a choice of alternative voting procedures when the range is constrained to be small, while only one remains as the only possibility in contexts where agents are a priori given more choices. On the other hand, Proposition 2 and Theorem 1 show why it is necessary to assume preferences with intervals in the tops. Proposition 2 and Theorem 1 show that, even considering "coordinated" voting procedures and assuming additive preferences, only dictatorial voting procedures with focus on $\mathcal{C}$ are strategy-proof and satisfactory. These findings illustrate that some voters may want to manipulate a "coordinated" voting procedure, since only preferences' tops restricted to feasible alternatives are taken into account by this class of voting procedures, and these might not be the voters' unrestricted tops (see Example 1).

Our introductory example (Section 1) shows that with additive preferences, well known strategy-proof voting procedures may fail to satisfy the feasibility restriction. Moreover, in this example unfeasible results were obtained even when each of the voters had preferences with an interval in the top. In order to obtain a feasible outcome, the voting procedure must "coordinate" the votes in order to rule out unwanted results. Thus, first we are going to define a coordinated voting procedure; but before that let us first introduce the following definition.

Definition 8 *Two nonempty families of nonempty winning coalitions, $\mathcal{W}$ and $\mathcal{W}'$, are mutually exclusive if and only if $W \in \mathcal{W}$ implies $N \backslash W \notin \mathcal{W}'$ and $W \in \mathcal{W}'$ implies $N \backslash W \notin \mathcal{W}$.*

In other words, this definition says that: two nonempty families of nonempty winning coalitions are mutually exclusive if, their corresponding minimal winning coalitions have at least one voter in common. This definition is useful to define a voting procedure that ensures feasibility. The reason for this is the following: a procedure that guarantees feasibility must enforce that whenever two disconnected intervals are chosen, then all the intervals that are in the middle of these two must also be chosen. Let us denote by $\mathcal{W}_{a_l}^M$ the nonempty family of minimal winning coalitions for a_l, $1 \leq l \leq k$.

Definition 9 *Let $f : \mathcal{P}^n \to 2^K$ be a voting by committees with focus on $\mathcal{C}$, then f is coordinated if the following two conditions are satisfied:*

(D9.1) $\mathcal{W}_{a_1}$ and $\mathcal{W}_{a_k}$ are mutually exclusive.

(D9.2) (i) For $k = 3$: for each pair (W_{a_1}, W_{a_3}), where $W_{a_1} \in \mathcal{W}_{a_1}^M$ and $W_{a_3} \in \mathcal{W}_{a_3}^M$, there is $W \in \mathcal{W}_{a_2}^M$ and $W \subseteq W_{a_1} \cap W_{a_3}$. (ii) For $k > 3$: given $W \subseteq \left[\underset{W \in \mathcal{W}_{a_1}^M}{\cap} W \right] \cap \left[\underset{W \in \mathcal{W}_{a_k}^M}{\cap} W \right]$, W is the unique minimal winning coalition for each a_l, where $1 < l < k$.

Proposition 1 *If $f : \mathcal{P}^n \to 2^K$ is coordinated then f is satisfactory.*

Notice that proposition 1 applies to all preferences, since voting procedures that are coordinated are voting by committees with focus on $\mathcal{C}$. The proof of this proposition is straightforward, since a coordinated voting procedure is a particular case of a voting procedure satisfying the intersection property proposed by Barberà et al. (1997); and is also a particular case of the intersection property by Nehring and Puppe (2007).

3.1 Additive Preferences

Proposition 2 *A voting procedure, $f : (\mathcal{P}^A)^n \to 2^K$, is coordinated and strategy proof on $\mathcal{P}^A$ if it is dictatorial with focus on $\mathcal{C}$.*

Let us illustrate this with an example (example 1). But first, we need to introduce the formal definition of voting by quota.

Definition 10 *A voting procedure, $f : \mathcal{P}^n \to 2^K$, is voting by quota if there exists Q_{a_l} between 1 and $\#N$ for each a_l such that for all profiles $P^n \in \mathcal{P}^n$, we have $a_l \in f(P^n)$ if and only if $\#\{i \mid a_l \in \tau_{\mathcal{C}}(P_i)\} \geq Q_{a_l}$.*

A particular case of voting by quota is unanimity, that is a voting by quota such that $Q_{a_l} = n$ for all $a_l \in K$. Notice that, if we consider unanimity then automatically (D9.1) and (D9.2) are satisfied. Thus, unanimity is a coordinated voting procedure.

Example 1 *Consider* $K = \{a_1, a_2, a_3\}$*, three voters (*1*,* 2 *and* 3*), the feasible alternatives are the intervals and* f *is the unanimity voting procedure. The preferences of voters* 1*,* 2 *and* 3 *are additive preferences such that:* $\tau_{2^K}(P_2) = \tau_{2^K}(P_3) = a_3$ *and* $\{a_1, a_3\}P_1\{a_1\}P_1\{a_3\}P_1\emptyset P_1\{a_1, a_2, a_3\}$*. This is manipulable because if voter* 1 *reports* P_1 *then* $f(P_1, P_2, P_3) = \emptyset$*; and if voter* 1 *reports* P_1'*, which is an additive preference such that* $\tau_{2^K}(P_1') = a_3$*, then* $f(P_1', P_2, P_3) = a_3$*.*

The previous example shows why proposition 2 is valid. Although, proposition 2 is implied by our main impossibility result, which is Theorem 1.

Theorem 1 *A voting procedure,* $f : (\mathcal{P}^A)^n \to 2^K$*, is satisfactory and strategy-proof on* $\mathcal{P}^A$ *if and only if it is a dictatorial with focus on* $\mathcal{C}$*.*

This result is a particular case of a theorem stated by Barberà et al. (2005) (see appendix A). In appendix B, we provide a proof of this theorem in order to illustrate the connection of our result with that of Barberà et al. Basically, the proof shows that given our special constraint, we can not decompose the range of the voting procedure into rules that choose in each dimension of a Cartesian product (this idea dates back to the work of Border and Jordan (1983)). In other words, in any possible decomposition of the range we found that if we let the voters to vote in each dimension of a Cartesian product then we obtain results that are not feasible or there are alternatives that are feasible but they can not be chosen.

3.2 Separable Preferences

Example 1 points out one important aspect of the formulation of our problem. It is the fact that the voting procedure considers preferences' tops restricted to intervals, what induces voter 1 to manipulate. Therefore, in order to obtain a possibility result, we need to introduce a new domain restriction where preferences have intervals in the top.

Theorem 2 *A voting procedure,* $f : (\bar{\mathcal{P}}^S)^n \to 2^K$*, is satisfactory and strategy-proof on* $\bar{\mathcal{P}}^S$ *if and only if it is coordinated.*

This theorem derives from the results presented in Barberà et al. (1997), and in Nehring and Puppe (2007). The proof is in appendix C. This theorem provides a collection of voting procedures that are strategy-proof when preferences are separable with intervals in the top. In the next section, we provide a particular application of this result; we characterize the class of anonymous voting by committees, which is voting by quota, when preferences are separable with intervals in the top.

4 Voting by Quota

The following examples illustrate the ideas that we are going to present in Proposition 3, which is a corollary of Theorem 2.

Example 2 *Consider $k = 3$ and f voting by quota. Notice, that the only alternative that is not an interval is $\{a_1, a_3\}$, so if a_1 and a_3 are chosen then we need to guarantee that a_2 is also chosen in order to obtain an interval. This can be achieved if we make certain that (D9.1) and (D9.2) are satisfied. Notice, that this is the case if the following conditions on quotas are satisfied: the sum of the quotas of a_1 and a_3 must be greater than the cardinality of N (this guarantees(D9.1)); and the quota for a_2 must not be greater than the sum of the quotas of a_1 and a_3, minus the cardinality of N (this ensures (D9.2)).*

Example 3 *Consider $k = 4$ and f voting by quota. In this case, the only way to obtain an interval from the voting procedure, is requiring that the quotas of a_1 and a_4 to be equal to the cardinality of N, and any quota for a_2 and a_3. Not requiring unanimity for a_1 or a_4 will lead us to sets that are not intervals. For instance, assume that the quotas of a_1 and a_4 are equal to $\#N - 1$. Then if we require that the quotas of a_2 and a_3 to be equal to 1, it is easy to find profiles such that the voting procedure choose a set that is not an interval; e.g. $\#N - 1$ voters have in the top of their preferences the set $\{a_1\}$ and the other voter has the set $\{a_3\}$.*

Notice that, when k is greater than 4, unanimity is the only possible voting by quota that guarantees that the outcome of the voting procedure is an interval. In order to illustrate this idea, consider $k = 5$. If $k = 5$ then there are three intervals that are not in the extremes of the line, they are a_2, a_3 and a_4. Requiring a quota equal to n for the extremes (i.e. for a_1 and a_5), as we have done in the previous example, and any quota less than n for the rest of the elements of K, does not guarantee feasibility. Since it is easy to find preferences such that, for instance, $\{a_2, a_4\}$ is chosen. Moreover, requiring quotas equal to n for all the elements of K, except for example quota 1 for a_3, does not ensure an interval as an outcome of the voting procedure. Therefore, when the cardinality of K is greater than four, the existence of at least three elements that are not in the extremes of the line introduces many possible infeasibilities that can not be ruled out by voting by quotas, except in case of unanimity.

Proposition 3 *A voting by quota, $f : (\bar{\mathcal{P}}^S)^n \to 2^K$, is a satisfactory voting procedure and is strategy proof on $\bar{\mathcal{P}}^S$ if and only if it satisfies one of the following conditions:*

(P3.1) If $k = 3$ then $Q_{a_1} + Q_{a_3} > \#N$ and $Q_{a_2} \leq [Q_{a_1} + Q_{a_3}] - \#N$.
(P3.2) If $k = 4$ then $Q_{a_1} = Q_{a_4} = \#N$ and $Q_{a_2} = Q_{a_3} \leq \#N$.
(P3.3) If $k \geq 5$ then $Q_{a_l} = \#N$ for all $a_l \in K$.

Since this result derives from theorem 2, we skip the proof. As we illustrated in our previous examples: (P3.1), (P3.2) and (P3.3) guarantee that (D9.1) and (D9.2) are satisfied. Unanimity is a special case in (P3.1) and in (P3.2), and is the unique voting procedure in (P3.3). Hence, this characterization result depends crucially on the number of choices that voters are presented with. This result shows that, there is a choice of alternative voting procedures when the range is constrained to be small, while unanimity rule remains as the only possibility in contexts where agents are a priori given more choices. Therefore, controlling for the number of alternatives can be a determinant tool for a designer.

5 Conclusions

We started this paper showing that there are many situations where a group of voters has to choose intervals, whose union must also be an interval. We modelled those situations as a voting procedure, where voters are asked to vote for sets of intervals, restricting the outcomes to intervals. In this paper, we study the existence of strategy-proof voting procedures in that context.

We obtained two kind of results: an impossibility result and a possibility result. The impossibility theorem shows that only dictatorial rules are strategy-proof when the domain is restricted to additive preferences. On the other hand, the possibility theorem characterizes a collection of strategy-proof voting procedures that choose intervals. The latter finding is under the assumption that preferences are separable with intervals in the tops, and that voting procedures coordinate the actions of the voters to enforce feasibility.

We also characterize a class of strategy-proof and anonymous voting procedures that only chooses intervals, i.e. voting by quota. In particular, we find that quotas must satisfy certain conditions in order to ensure a feasible result. These conditions imply that when the number of alternatives increases, the class of voting by quota that respects feasibility shrinks. That is, unanimity is the only voting by quota that respects feasibility when more than four nonoverlapping intervals are presented to the voters. This is important for many reasons. First, unanimity is a rule used in many societies. Hence, this provides a justification for the use of this rule. Second, this result draws attention to the design of the voting procedure. In case the designer would like to choose between alternative voting procedures, then our result implies that few alternatives need to be considered for voting. Finally, in case that the designer wants to give some power to minorities, in the sense that few voters could impose a result (this could be done by requiring a quota different from the total number of voters for a given interval), then few choices must be presented to the voters.

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Appendix A. A Useful Result

Before presenting the Theorem, let us state some definitions that are used in the statement of this result. The aim of the following definitions is to provide a methodology to decompose the range of a voting scheme into a Cartesian Product. First, consider the following notation: we denote by R_f the set of chosen intervals, i.e. $R_f = \{a_l \in K \mid a_l \in A \text{ for some } A \in \mathcal{R}_f\}$.

Definition 11 *Given a voting procedure* $f : \mathcal{P}^n \to 2^K$ *and* $A \subseteq R_f$*, the active components of* A *are defined by* $\mathcal{AC}(A) = \{B \cap A \mid B \in \mathcal{R}_f\}$.[7]

[7] Notice that $\mathcal{AC}(A)$ is a function of f, but for simplicity, in the notation we suppress this dependence.

Example 4 *Consider $k = 3$, so we have $K = \{a_1, a_2, a_3\}$, and f a satisfactory voting procedure. This defines the following set of feasible alternatives $\{\emptyset, \{a_1\}, \{a_2\}, \{a_3\}, \{a_1, a_2\}, \{a_2, a_3\}, \{a_1, a_2, a_3\}\}$. The active components of the sets $\{a_2\}$ and $\{a_2, a_3\}$ are $\mathcal{AC}(\{a_2\}) = \{\emptyset, \{a_2\}\}$, and $\mathcal{AC}(\{a_2, a_3\}) = \{\emptyset, \{a_2\}, \{a_3\}, \{a_2, a_3\}\}$.*

Definition 12 *Given $A \subseteq B \subseteq R_f$ the range complement of A relative to B is defined by $\mathcal{C}_f^B(A) = \{C \subseteq R_f \setminus B \mid A \cup C \in \mathcal{R}_f\}$.*

Example 5 *(Continuation of Example 4) The range complement of the sets $\{a_2\}$ and $\{a_2, a_3\}$ relative to $\{a_2, a_3\}$ are $\mathcal{C}_f^{\{a_2,a_3\}}(\{a_2\}) = \mathcal{C}_f^{\{a_2,a_3\}}(\{a_2, a_3\}) = \{\{a_1\}\}$.*

Definition 13 *$A \subseteq K$ is a section of R_f if for all active components $B, C \in \mathcal{AC}(A)$ we have $\mathcal{C}_f^A(B) = \mathcal{C}_f^A(C)$.*

Example 6 *(Continuation of Example 4) Neither $\{a_2\}$ nor $\{a_2, a_3\}$ is a section, since $\mathcal{AC}(\{a_2\}) = \{\emptyset, \{a_2\}\}$ and $\mathcal{C}_f^{\{a_2\}}(\emptyset) = \{\{a_1\}, \{a_3\}\} \neq \mathcal{C}_f^{\{a_2\}}(\{a_2\}) = \{\{a_1\}, \{a_3\}, \{a_1, a_3\}\}$; and since $\mathcal{AC}(\{a_2, a_3\}) = \{\emptyset, \{a_2\}, \{a_3\}, \{a_2, a_3\}\}$ and $\mathcal{C}_f^{\{a_2,a_3\}}(\emptyset) = \mathcal{C}_f^{\{a_2,a_3\}}(\{a_2\}) = \mathcal{C}_f^{\{a_2,a_3\}}(\{a_2, a_3\}) \neq \mathcal{C}_f^{\{a_2,a_3\}}(\{a_3\})$.*

Definition 14 *A partition $\{A_1, ..., A_n\}$ of R_f is a Cartesian decomposition of R_f if for all $q = 1, ..., n$, A_q is a section of R_f. A Cartesian decomposition is called minimal of there is no finer Cartesian decomposition of R_f.*

Example 7 *(Continuation of Example 4) The partition $\{\{a_1, a_2, a_3\}\}$ is the minimal Cartesian decomposition of R_f.*

Now, we are in position to state the Theorem.

Theorem 3 *(Barberà, et al. (2005)) A voting procedure $f : (\mathcal{P}^A)^n \to 2^K$ is strategy-proof if and only if it is voting by committees with focus on $\mathcal{R}_f$ and with the following properties:*

(T3.1) $\mathcal{W}_{a_l}$ and $\mathcal{W}_{a_m}$ are equal for all a_l and a_m in the same active component of any section with two active components in R_f's minimal Cartesian decomposition,

(T3.2) $\mathcal{W}_{a_l}$ and $\mathcal{W}_{a_m}$ are mutually exclusive for all a_l and a_m in different active components of the same section in R_f's minimal Cartesian decomposition, when there are only two active components in this section, and

(T3.3) $\mathcal{W}_{a_l}$ is dictatorial with focus on $\mathcal{R}_f$ and equal for all a_l's in the same section in R_f's minimal Cartesian decomposition, when this section has more than two active components.

Appendix B. Proof of Theorem 1

Proof. Consider a partition $K = \{a_1, ..., a_k\}$ of $\mathcal{A}$ with $k > 2$, and a satisfactory voting procedure $f : (\mathcal{P}^A)^n \to 2^K$.

First, we are going to show that $\{a_1, ..., a_k\}$ is a section. Notice that, $\mathcal{AC}(\{a_1, ..., a_k\}) = \mathcal{R}_f$ and since $R_f \setminus \{a_1, ..., a_k\} = \emptyset$, then $\mathcal{C}_f^{\{a_1,...,a_k\}}(A)$ is the same for all $A \in \mathcal{R}_f$. Therefore, $\{a_1, ..., a_k\}$ is a section.

Second, we are going to prove that when $k = 3$, $\{a_1, a_2, a_3\}$ is the minimal Cartesian decomposition of R_f. Consider any non empty $A \subset K$, such that $A \neq \{a_1, a_3\}$. Notice that $\{a_1, a_3\}$ is a section and that $\emptyset \in \mathcal{AC}(A)$.Then, take two intervals $a_l \in R_f \setminus A$ and $a_m \in A$. Then we have two cases: (i) if $|l - m| = 1$, then there exists $a_n \in R_f \setminus A$ such that $|n - m| = 1$ and $a_n \neq a_l$, where $\{a_l, a_n\} \in \mathcal{C}_f^A(\{a_m\})$ but $\{a_l, a_n\} \notin \mathcal{C}_f^A(\emptyset)$; (ii) and if $|l - m| > 1$, then $\{a_l\} \in \mathcal{C}_f^A(\emptyset)$ but $\{a_l\} \notin \mathcal{C}_f^A(\{a_m\})$. Hence, A is not a section.

Finally, we are going to show that any non empty $A \subset K$ is not a section when $k > 3$. Consider two intervals $a_l \in R_f \setminus A$ and $a_m \in A$, such that $|l - m| > 1$. Notice that $\emptyset, \{a_m\} \in \mathcal{AC}(A)$. Then $\{a_l\} \in \mathcal{C}_f^A(\emptyset)$ but $\{a_l\} \notin \mathcal{C}_f^A(\{a_m\})$. Hence, A is not a section.

Therefore, we have that $\{a_1, ..., a_k\}$ is the minimal Cartesian decomposition of R_f, then by (T3.3) $f : (\mathcal{P}^A)^n \rightarrow 2^K$ is dictatorial. ■

Appendix C. Proof of Theorem 2

Theorem 2 follows from Propositions 5 and 6 (below).

Proposition 4 *(Barberà, et al. (1991)) If a voting procedure $f : (\mathcal{P}^S)^n \rightarrow 2^K$ is strategy-proof on $\mathcal{P}^S$ and satisfies voter sovereignty on 2^K if and only if f is voting by committees with focus on 2^K.*

In Proposition 4, we consider that f is voting by committees with focus on 2^K, since in Barberà et al. (1991) voting by committees is defined such that voters vote for their top alternative. From Proposition 4 we can derive Proposition 5, since all the preferences used to prove Proposition 4 can be restricted to preferences in $\bar{\mathcal{P}}^S$ without altering the rest of the proof (see Lemma 1 and Lemma 2 in Barberà et al. (1991)).

Proposition 5 *If a voting procedure $f : (\bar{\mathcal{P}}^S)^n \rightarrow 2^K$ is strategy-proof on $\bar{\mathcal{P}}^S$ and satisfies voter sovereignty on 2^K if and only if f is voting by committees with focus on $\mathcal{C}$.*

Proposition 6 *Let $f : (\bar{\mathcal{P}}^S)^n \rightarrow 2^K$ be a voting by committees with focus on $\mathcal{C}$. Then, f is satisfactory if and only if f is coordinated.*

Proof. First notice that, since f is voting by committees with focus on $\mathcal{C}$, then satisfies voter sovereignty on 2^K; and in particular, satisfies voter sovereignty on $\mathcal{C}$.

Let's start by proving that (D9.1) and (D9.2) are *sufficient conditions.*

Voter's sovereignty on $\mathcal{C}$ implies that $A \in f(\mathcal{C}^n)$, for every $A \in \mathcal{C}$. We must show that $f(P^n) \in \mathcal{C}$ for every $P^n \in \bar{\mathcal{P}}^S$.

Suppose that f satisfies (D9.1) and (D9.2), and assume that there exists $A \notin \mathcal{C}$ and $P^n \in \bar{\mathcal{P}}^S$ such that $f(P^n) = A$. Since f is voting by committees and satisfies (D9.1) and (D9.2), then f is coordinated. $A \notin \mathcal{C}$ implies that there exists $a_l \notin A$ such that: $a_m, a_n \in A$ and $m < l < n$. By (D9.1) and (D9.2), we know that W_{a_l} a minimal winning coalition for a_l, satisfies $W_{a_l} \subseteq W_{a_m} \cap W_{a_n}$; where W_{a_m} and W_{a_n} are minimal winning coalitions for a_m and a_n, respectively. Since $\tau_{2^K}(P_i) \in \mathcal{C}$ for each $i \in N$, and $a_m, a_n \in A$ means that $a_m, a_n \in \tau_{2^K}(P_i)$ for each $i \in W_{a_m} \cap W_{a_n}$; then $W_{a_l} \subseteq W_{a_m} \cap W_{a_n}$ implies that $a_l \in A$, i.e. $A \in \mathcal{C}$. Contradiction.

The (D9.1) and (D9.2) are *necessary conditions.*

Suppose that f does not satisfy (D9.1) and (D9.2). We must show that there exists $P^n \in \bar{\mathcal{P}}^S$ such that $f(P^n) \notin \mathcal{C}$. Consider the case that f does not satisfy (D9.1), i.e., there exists W_{a_1} a minimal winning for a_1 and there exists W_{a_k} a minimal winning coalition for a_k, such that $W_{a_1} \cap W_{a_k} = \emptyset$. Assume that $P^n \in \bar{\mathcal{P}}^S$ is such that: $\tau_{2^K}(P_i) = \{a_1\}$ for each $i \in W_{a_1}$, $\tau_{2^K}(P_i) = \{a_k\}$ for each $i \in W_{a_k}$, and $W_{a_1} \cup W_{a_k} = N$. Then, $f(P^n) = \{a_1, a_k\} \notin \mathcal{C}$ (a similar argument applies if (D9.2) is not satisfied). ■