Repeated Games Played in a Network

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Abstract

Delayed perfect monitoring in an infinitely repeated discounted game is modelled by allocating the players to a connected and undirected network. Players observe their immediate neighbors' behavior only, but communicate over time the repeated game's history truthfully throughout the network. The Folk Theorem extends to this setup, although for a range of discount factors strictly below 1, the set of sequential equilibria and the corresponding payoff set are reduced under certain conditions. A general class of games is analyzed without imposing restrictions on the dimensionality of the payoff space. A network result obtains; namely, the level of cooperation in this setup depends on the network's diameter, and not on its clustering coefficient as in other models. This model applies in more general situations (without network) in which there is a heterogeneous delay between information generation and the players' reaction to it.

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1 Introduction

Repeated games capture dynamic strategic interaction between impatient economic agents. Usually, there are equilibria that do not arise in a one-shot game. The associated payoff vectors, moreover, can be Pareto superior to the ones achieved in all stage game equilibria. The well-known Folk Theorem states this result. This paper focuses on infinitely repeated discounted games for which Fudenberg, Levine and Takahashi (2007), thereafter FLT, obtain the subgame-perfect Folk Theorem. They dispose of any dimensionality condition previously imposed, for example, by Abreu, Dutta and Smith (1994), thereafter ADS, and extend the result of Wen (1994) to (unobservable) mixed actions.

For simplicity, other strong assumptions are made, such as that a player observes his opponents' behavior immediately and perfectly. This assumption is relaxed in the imperfect monitoring literature, in which each player receives an imperfect private or public signal of the action profile played. Under certain conditions, the set of sequential equilibria, or of other equilibrium concepts that extend subgame-perfectness to games of imperfect information, is non-empty. In some cases even the Folk Theorem obtains. The interested reader is referred to a private monitoring survey by Kandori (2002) and Mailath and Samuelson's (2006) textbook.

The aim of this paper is to model delayed perfect monitoring by allocating the players, that play an infinitely repeated discounted game, to a connected and undirected network. In each period, a player observes his neighbors' action choices and communicates truthfully these observations and other information he has received before to all neighbors. The players thus take decisions under imperfect information in any but the first period and the concept of sequential equilibrium is used. Nevertheless, the entire history of the repeated game spreads gradually throughout the network over time. The network gives a structure to this heterogeneous flow of information. It is also possible to interpret the delay in information transmission as being due to the time it takes a player to process information or to react to new information.[1]

In reality, an impatient economic agent frequently participates in a repeated multi-player game. He only observes directly the behavior of few other participants. However, over time he gets to know what any other player did and which influence this had on his payoff. Then, he can react to it. In many industries, such as the car industry, the network is given by contractual relationships between companies that are organized along

[1] The network is equivalent to a matrix of dimension "number of players" times "number of players", in which each entry specifies the delay with which the corresponding two players obtain information about each other. All results hold as well when this matrix is asymmetric, or equivalently, for directed networks.

the value chain. The behavior of any participant in the industry has an influence on the outcome of all other participants. For example, a raw material producer might face a disruption of his works or an increase in his cost. Although this affects all companies in the industry, they only get to know this after some delay, namely, once the information about it flows throughout the network, for example, in form of an increase in prices. Their payoff (in net present value) is affected immediately and they would respond to this if they had the corresponding information already. Also the big firm at the center of the network, by enforcing changes on its suppliers, triggers such flows of information which affect the payoff of any network member.

However, this model is more general and the network is only a mean to visualize delay and the players' heterogeneous reaction to new information. A real example is taken from the mobile phone industry. In 2000, a semiconductor plant of a supplier to both Nokia and Ericsson was affected by a fire. Initially, the damage was thought to be minor while it turned out after a short while that the plant would be closed for months. While Nokia steadily gathered information and reacted to it immediately, Ericsson did not do this after it received the first notice which was optimistic. As a consequence, Nokia locked in all substitute supplies while Ericsson was deprived of one of its main inputs, mobile phone chips, for several months. In terms of this model, Nokia is placed closer to the semiconductor plant while Ericsson receives the information with some delay.[2] The strategic interaction between Nokia, Ericsson and the semiconductor plant can be viewed as a repeated game in which each player's payoff is affected by all players' decisions. The semiconductor plant's failure to deliver memory chips is then a unilateral deviation.

Under the assumption of truthtelling, the Folk Theorem extends to the delayed perfect monitoring model, that is, any feasible and strictly individually rational payoff vector can be supported by a sequential equilibrium strategy profile when the players are sufficiently patient. Then, they do not mind to receive the repeated game's history of action profiles gradually over time. However, for a range of discount factors strictly below 1, the delay in information transmission caused by the network may trigger a player's deviation from some previously agreed sequence of play. The reduction in the set of sequential equilibria in comparison to the perfect monitoring case, which arises when players are impatient, seems to reflect many real situations well. Moreover, due to the network, the equilibrium payoff space is reduced for a range of discount factors and a strategy profile is a sequential equilibrium for a smaller range of discount factors.

[2] Similarly, Cho and Li (1999) model infinitely repeated undiscounted games with two players and two actions. Each player is an organization modelled as a hierarchical network which processes information. This captures bounded rationality and the corresponding cost of complexity are studied.

The concept of punishment reward is adapted to the network case and in order to analyze a general class of games, no restriction is imposed on the dimensionality of the payoff space. Frequently, in this literature, the simplifying assumption is made that there is no pair of players whose payoff is "perfectly correlated". In reality, players may have a common interest and the payoff of each increases and decreases proportionally with the other's. This possibility is considered explicitly here. Whereas in a perfect monitoring model this assumption is purely technical and may be disposed of (see section 2.3.1), in this setup it makes a substantial difference due to the heterogeneous information flow. As a consequence, the *effective minmax* concept has to be used.

This model also contributes to the network literature in which the clustering coefficient, or similar measures of local connectedness, usually determine the level of cooperation sustainable in a network. Conversely, in this model, the network's diameter is decisive since it determines the delay after which all players in the network have the same information about all others. The two measures are not related as is illustrated in an example.

In the related literature there are different setups. In one, all players play the same repeated game and a player observes an imperfect private or public signal of each action profile played.[3] Sometimes, imperfect information is resolved by public announcements (see section 6.2 for references). In other models, each pair of neighbors in a network plays a bilateral repeated game, usually the Prisoner's Dilemma. A player observes only his neighbors' behavior and communicates with the same group.[4]

Renault and Tomala (1998) analyze a setup as in this paper. All players participate in the same repeated game and a bilateral communication and observation network exists, but a player never reports his own action choice. They study uniform Nash Equilibria—which is much weaker than sequential equilibrium—of finitely and infinitely repeated undiscounted games when the players form a 2-connected and directed graph. Since this implies that two distinct paths between any pair of players exist, lies are prevented in equilibrium. However, the payoff accumulation stops during a communication phase.

Haag and Lagunoff (2006) let each neighborhood play an infinitely repeated discounted Prisoner's Dilemma. They analyze the optimal network structure when each player's discount factor is randomly determined before the game begins. Ben-Porath and Kahneman (1996) study sequential equilibria of infinitely repeated discounted games in which the players form a (not necessarily connected) network. They assume that players publicly

[3] Fudenberg, Levine and Maskin (1994) model imperfect public monitoring and Kandori (2002) surveys the imperfect private monitoring literature.

[4] Examples for this are Lippert and Spagnolo (2005) and Vega-Redondo, Marsili and Slanina (2005). Both are discussed in more detail in section 5.4. In Kandori (1992) and Ellison (1994), each pair of players that plays a repeated Prisoner's Dilemma is randomly and repeatedly rematched.

announce their own action choices and observations made about their neighbors in a strategic way, that is, including lies. When each group contains strictly more than two players unilateral deviations are detectable, and hence, do not occur in equilibrium. In Ben-Porath and Kahneman (2003) this idea is extended. Since monitoring is costly, only one monitor is assigned to every player. After an incompatible announcement, which in equilibrium does not occur, both players are punished and the monitor is substituted.

In a companion paper of this, Kinateder (2008) analyzes the repeated network Prisoner's Dilemma. Due to its special structure, additional results obtain. The reduction in the equilibrium payoff vector set is depicted graphically and formal results under strategic communication are given.

The next section introduces notation and definitions. In section 3, the model's features are illustrated for the Prisoner's Dilemma. This part of the paper partly coincides with the corresponding section in Kinateder (2008). In section 4, the information sharing process and the punishment reward phase are defined. Both are prerequisites for the Folk Theorem, which is stated in section 5, along with conditions under which impatient players deviate from a given sequence of action profiles. In the same section, moreover, other strategy profiles which yield the Folk Theorem and the model's relation with the network literature are discussed. The model is presented in pure actions. Before concluding, remarks about its extension to mixed actions and to strategic communication follow.

2 Preliminaries

2.1 Stage Game and Network

Each player i in the finite set of players $I = \{1, ..., n\}$ has a finite and non-empty set of pure actions A_i; a pure action a_i is an element of this set. The pure action space of the stage game is $A = \times_{i \in I} A_i$, with generic element a, called pure action profile. To emphasize player i's role, a is written as (a_i, a_{-i}). For any non-empty set of players $S \subset I$, let $A_S = \times_{i \in S} A_i$, and denote by a_S an element of this set. Player i's payoff function is a mapping $h_i : A \to \mathbb{R}$, and the payoff function $h : A \to \mathbb{R}^n$ assigns a payoff vector to each pure action profile. The stage game in normal form is then the tuple $G \equiv (I, (A_i)_{i \in I}, (h_i)_{i \in I})$. Finally, define the convex hull of the finite set of payoff vectors corresponding to pure action profiles in G as $co(G) = co\{x \in \mathbb{R}^n \mid \exists\, a \in A : h(a) = x\}$.

The players in set I are the vertices of a network g, whose graph is defined as the pair (I, E), where $E \subseteq I \times I$ denotes the set of links between them. A link from player i to player j is denoted by (i, j). Graph (I, E) is undirected, that is, for all $i, j \in I$, (i, j) if, and only if, (j, i). Given network g, a path between two distinct players i and j is defined as a sequence of distinct players $i_1, ..., i_r$ such that $i_1 = i$, $i_r = j$, and $(i_{l-1}, i_l) \in E$, for all $1 < l \leq r$. Its length is $r - 1$. Network g is assumed to be connected. Hence, each player is connected to at least one other player directly and to all others via paths of finite lengths. The length of the shortest path between two distinct players i and j is called *distance* between i and j and is denoted by d_{ij}. The *largest distance* between player i and any other player in the network is defined by $d_i = \max_{j \in I} d_{ij}$, and the *diameter* of network g is the maximal largest distance among all players, that is, $d = \max_{i \in I} d_i$. Finally, denote player i's set of direct neighbors by $i(1) = \{j \in I \mid d_{ij} = 1\}$, and for any $2 \leq m \leq d_i$, define his set of *m-neighbors* as $i(m) = \{j \in I \mid d_{ij} = m\}$.

When the stage game is played repeatedly, in each period, a player first chooses an action, in a way specified below, and then makes observations and communicates with his neighbors. He observes the actions chosen by his immediate neighbors, before receiving the information they received one period earlier from their neighbors. Similarly, he reveals to any direct neighbor the action he plays, before communicating him the information he received one period ago. Hence, information flows one link per period and with a $d_i - 1$ period lag, player i gets to know the repeated game's entire history.[5] It is assumed that players always truthfully reveal what their neighbors did and told them. How to extend this model to strategic communication is discussed in section 6.2.

Additionally, a player has perfect recall. Hence, for any player $i \in I$ at any $t \geq 1$, there is a *set of observations*, denoted by Ob_i^t, that includes all histories of observations that player i may have made at the end of period t. It is defined recursively as

$$\begin{array}{rcl} Ob_i^1 & = & A_i \times A_{i(1)}, \\ Ob_i^2 & = & A_i^2 \times A_{i(1)}^2 \times A_{i(2)}, \\ \vdots & \vdots & \vdots \\ Ob_i^t & = & A_i^t \times A_{i(1)}^t \times A_{i(2)}^{t-1} \times \cdots \times A_{i(d_i)}^{t-d_i+1} \end{array}$$

for all $t \geq d_i$, where for any $1 \leq m \leq d_i$ and any $t \geq 1$, $A_{i(m)}^t = (\times_{j \in i(m)} A_j)^t$. Player i's observation at t is denoted by $ob_i^t \in Ob_i^t$. Given G and g, a sequence of action profiles $\{a^t\}_{t=1}^{\infty}$, where $a^t \in A$ for all $t \geq 1$, generates a sequence of observations for player i,[6]

[5] At the end of any $t \geq d_i$, for example, player i knows the actions played in period t by himself and all players in $i(1)$, the actions played by himself and all players in $i(1)$ and $i(2)$ at $t - 1, ...,$ and finally the actions played by all players at $t - d_i + 1$ and at any point in time before.

[6] Equivalently, this setup can be interpreted as follows. Each action profile a^t generates a public signal with a delay of $d - 1$ periods and certain private signals in all periods s, where $t \leq s < t + d - 1$.

$$
\begin{aligned}
ob_i^1 &= (a_i^1, a_{i(1)}^1), \\
ob_i^2 &= (a_i^1, a_{i(1)}^1, a_{i(2)}^1, a_i^2, a_{i(1)}^2), \\
\vdots &\;\;\vdots\;\;\vdots \\
ob_i^t &= (\{a_i^s\}_{s=1}^t, \{a_{i(1)}^s\}_{s=1}^t, \{a_{i(2)}^s\}_{s=1}^{t-1}, ..., \{a_{i(d_i)}^s\}_{s=1}^{t-d_i+1}\})
\end{aligned}
$$

for all $t \geq d_i$. At any $1 \leq t < d_i$, player i is not yet informed about the behavior of at least one other player. At $t = d_i$, $ob_i^{d_i}$ contains the actions chosen by all players at $t = 1$. Abusing notation, this is referred to as $a^1 \in ob_i^{d_i}$ (since a^1 belongs to A). At any $t > d_i$, the action profiles $a^1, ..., a^{t-d_i+1}$ are identified by player i, and hence, in an abuse of terminology, said to be elements of ob_i^t. Thus, at any $t \geq 1$, the sequence of action profiles generates an observation profile $ob^t \in Ob^t$, where $Ob^t = \times_{i \in I} Ob_i^t$. The players organized in this way play an infinitely repeated discounted game.

2.2 Repeated Game with Delayed Perfect Monitoring

In the infinitely repeated discounted game played on the fixed network g, thereafter called repeated network game, at each point in discrete time, $t = 1, 2, ...$, stage game G is played. Set I contains at least three players since otherwise the network analysis is trivial.

Let player i's set of strategies be $F_i = \{\{f_i^t\}_{t=1}^\infty \mid f_i^1 \in A_i$, and for all $t > 1$, $f_i^t : Ob_i^{t-1} \to A_i\}$. At any $t \geq 1$, player i's strategy $f_i = \{f_i^t\}_{t=1}^\infty$ prescribes him to choose an action. For $t > 1$, this is a mapping from his *set of observations* to his action set. The cartesian product of all players' strategy sets $F = \times_{i \in I} F_i$, constitutes the strategy space of the repeated network game. A strategy profile $f = (f_1, ..., f_n)$ is an element of F. To emphasize player i's role, it is written as (f_i, f_{-i}). At any $t \geq 1$, each $f \in F$ recursively generates a pure action profile $a^t(f) = (a_1^t(f), ..., a_n^t(f))$ and a corresponding observation profile $ob^t(f) = (ob_1^t(f), ..., ob_n^t(f))$: for any player i, let $a_i^1(f) = f_i^1$ and $ob_i^1(f) = (a_i^1(f), a_{i(1)}^1(f))$, and for $t > 1$ given $ob_i^{t-1}(f) \in Ob_i^{t-1}$, $a_i^t(f) = f_i^t(ob_i^{t-1}(f))$ and $ob_i^t(f)$ is defined accordingly. Each $f \in F$ thus generates a sequence of action profiles $\{a^t(f)\}_{t=1}^\infty$, which in turn generates a sequence of observation profiles $\{ob^t(f)\}_{t=1}^\infty$.

Given a common discount factor $\delta \in [0, 1)$,[7] the function $H^\delta : F \to \mathbb{R}^n$ assigns a payoff vector to each strategy profile. Given $f \in F$, player i's payoff, $H_i^\delta(f) = (1 - \delta) \sum_{t=1}^\infty \delta^{t-1} h_i(a^t(f))$, is the $(1 - \delta)$-normalized discounted sum of stage game payoffs. The repeated network game associated with stage game G, discount factor δ and network g is then defined as the normal form game $G^{g,\delta} \equiv (I, (F_i)_{i \in I}, (H_i^\delta)_{i \in I})$.

[7] If the discount factor is interpreted as the probability with which the game is played again in the next period, then the probability that the repeated network game ends at T converges to 1 as T goes to infinity.

When g is complete, $i(1) = I \setminus \{i\}$ for all $i \in I$ and $G^{g,\delta}$ is identical to the infinitely repeated discounted game, referred to as G^{δ}. In this case, f_i simplifies: for any $t > 1$ it is a mapping from $A^{t-1} = (\times_{i \in I} A_i)^{t-1}$ to A_i, that is, each player conditions his action choice on the history of action profiles chosen by all players between periods 1 and $t-1$.

Moreover, the players have common knowledge of the game played, the form of the network[8] and the strategy choices available to all players. Finally and importantly, each player i observes his payoff with a delay of $d_i - 1$ periods. This prevents him from deducing other players' behavior by observing his payoff. At any $t \geq d_i$, however, player i knows the action profiles played between periods 1 and $t - d_i + 1$, and hence, he can calculate or equivalently observe his payoff for all these periods. If a player observes his or all players' payoffs immediately, this is a private or public signal, respectively, which moreover, may be imperfect. Both kinds of signal are already extensively studied in the literature on repeated games, as mentioned in the introduction, and it is straightforward to extend the results obtained in this way. The additional information the payoff may reveal is ignored here and a player is assumed to react only to information obtained via the network.

2.3 Payoff Vectors Generated by Sequential Equilibria

2.3.1 Individual Rationality without a Full-Dimensional Payoff Space

A player's individually rational payoff is the lowest to which he can be forced in a stage game. It obtains when he maximizes his payoff while all other players minimize it and is called *minmax* payoff. For any player $i \in I$, let his *minmax* payoff in pure actions be

$$\bar{\nu}_i \equiv \min_{a_{-i} \in A_{-i}} \max_{a_i \in A_i} h_i(a_i, a_{-i}). \tag{1}$$

ADS use the *minmax* payoff to define a player's individually rational payoff in any repeated (network) game,[9] in which the dimension of the payoff space is equal to the number of players, or at most of one dimension less. They show that this dimensionality condition holds whenever no two players have equivalent payoff functions in the corresponding stage game. Such games fulfill the NEU-condition of non-equivalent utilities.[10]

Two distinct players i and j have equivalent utilities (EU), when one player's payoff

[8] In most cases, common knowledge of the network is not required for the results obtained.

[9] All results mentioned in this section extend without loss of generality to the repeated network game.

[10] Two players have non-equivalent utilities when the payoff space's projection on the corresponding two player plane yields an ellipse or a negatively-sloped line. A positively-sloped line arises when one player's payoff increases monotonically in the other's. A stage game fulfills the NEU-condition if no pair of players' payoff space is a positively-sloped line and at most for one it is a negatively-sloped line.

function is a positive affine transformation of the other's, that is, there are $\alpha > 0$ and $\beta \in \mathbb{R}$ such that for all $a \in A$,

$$h_i(a) \;=\; \alpha h_j(a) + \beta. \tag{2}$$

This relation between EU-players i and j is denoted by $i \sim j$. When (2) is violated for two distinct players i and j, they have non-equivalent utilities, denoted by $i \nsim j$. The EU-players are partitioned into U sets, $S_1, ..., S_U$, such that $i \sim j$ holds for all $i, j \in S_u$, $1 \leq u \leq U$. Let $S \equiv \cup_{u \in U} S_u$, then $i \nsim j$ holds for all $i \in S_u$, $j \in S_{u'}$ such that $u \neq u'$, and for all $i \notin S$, $j \in I \setminus \{i\}$. Finally, assume that no player is universally indifferent among all action profiles, that is, for all $i \in I$, there are a, $a' \in A$ such that $h_i(a) \neq h_i(a')$.

When a stage game does not fulfill the NEU-condition, that is, $S \neq \emptyset$, a player's *effective minmax* payoff is his individually rational payoff. Following Wen (1994),[11] the *effective minmax* payoff in pure actions of any player $i \in S_u$ is defined as

$$\nu_i \;\equiv\; \min_{a \in A} \max\{h_i(a_j, a_{-j}) \mid j \in S_u,\; a_j \in A_j\}. \tag{3}$$

In each EU-group, a reference player is selected whose maximization yields any group member who is minimized the largest possible payoff. Thus, an EU-player's *effective minmax* payoff is not smaller than his *minmax* payoff, while for all other players $i \notin S$ the two payoffs are identical.

Denote the vector of *effective minmax* payoffs in pure actions by ν, and the pure action profile forcing player i to his *effective minmax* payoff by $\bar{a}^i$. It is one solution to the optimization problem on the right-hand-side of (3), on which the players agreed. Without loss of generality the *effective minmax* payoff of all players is normalized to 0, that is, for all $i \in I$, $h_i(\bar{a}^i) \equiv 0$. All players with equivalent utility to i's obtain a payoff of 0 as well when he is forced to his *effective minmax* payoff.

In a perfect monitoring model, the decisions of all players in an EU-group are identical since they are based on the same information, the commonly observed history of the repeated game. Hence, one player could replace the entire group. Conversely, in the repeated network game, each member of an EU-group chooses an action based on the observations he made thus far, and usually, these do not coincide.

[11] Wen (1994) defines the *effective minmax* concept in mixed actions, assuming however, that a player's deviation within the support of his mixed action is observable to the other players. For the general case, which includes unobservable deviations from mixed actions, this concept is defined by FLT.

2.3.2 Feasibility of Payoff Vectors and Sequential Equilibrium

The *set of feasible payoff vectors* of the repeated (network) game is defined as[12]

$$\mathcal{F} = \{x \in \mathbb{R}^n \mid \exists \; \{a^t\}_{t=1}^{\infty} : \; \forall \; t \geq 1, \; a^t \in A, \; \text{and} \; \forall \; i \in I, \; x_i = (1-\delta) \sum_{t=1}^{\infty} \delta^{t-1} h_i(a^t)\}.$$

Any feasible payoff vector is achievable by a sequence of pure action profiles.

The *set of feasible and strictly individually rational payoff vectors* is denoted by $\mathcal{F}^*$. It contains all feasible payoff vectors that are larger than $\nu = (0, ..., 0)$ and is defined as

$$\mathcal{F}^* \;\; = \;\; \mathcal{F} \;\; \cap \;\; \{x \in \mathbb{R}^n \mid x > \nu\}.$$

Any payoff vector in this set is a candidate to be supported by a sequential equilibrium.

A sequential equilibrium requires a strategy profile and a system of beliefs to be sequentially rational and consistent, respectively. In the repeated network game, the attention is restricted to a class of strategy profiles in which each player conditions his action choices only on his observations—he believes what he observes. In this class, each sequential equilibrium strategy profile is sequentially rational for any belief a player may have about the yet unobserved actions chosen by all other players in the most recent periods. Hence, beliefs are not modelled explicitly and a sequential equilibrium is said to exist when the condition of sequential rationality is fulfilled.

Definition 1. *A strategy profile $\dot{f} \in F$ is a* sequential equilibrium *(SE) of $G^{g,\delta}$, if for all $t \geq 1$ and given any $ob^t \in Ob^t$, $\{\dot{f}^{\tau}(ob^{\tau-1})\}_{\tau=t+1}^{\infty}$ is such that for all $i \in I$ and all $f_i \in F_i$,*

$$(1-\delta) \sum_{s=t+1}^{\infty} \delta^{s-1} h_i(a^s(\dot{f})) \;\; \geq \;\; (1-\delta) \sum_{s=t+1}^{\infty} \delta^{s-1} h_i(a^s(f_i, \dot{f}_{-i})).$$

When g is complete this definition includes G^{δ} and the concepts of sequential and subgame-perfect equilibrium coincide. However, equilibria of $G^{g,\delta}$ and G^{δ} are called sequential when Definition 1 is satisfied, and the corresponding sets of SE strategy profiles are denoted by $SE(G^{g,\delta})$ and $SE(G^{\delta})$, respectively. A strategy profile is a SE if, and only if, any player's finite unilateral deviation at any point in time is not profitable.[13]

[12] Any payoff vector in $co(\hat{G})$ is feasible for $\delta \in (1-\frac{1}{z}, 1)$, where z is the number of vertices of $co(\hat{G})$. For any discount factor in this range, sets $\mathcal{F}$ and $co(\hat{G})$ coincide; see Fudenberg, Levine and Maskin (1994).

[13] Since $\delta < 1$, a player's gain from a deviation of infinite length can be approximated by that of a finite deviation. Therefore, unilateral deviations of finite length from a strategy profile are not profitable if, and only if, it is a SE of the repeated network game; see Mailath and Samuelson (2006).

3 The Network makes a difference

The following example illustrates how imposing a network on a set of players affects the set of SE of a repeated game and partly coincides with the corresponding section in Kinateder (2008).[14] Let $\hat{G} = (I, A, h)$ be a generalized Prisoner's Dilemma game, where $n > 2$. At each point in time, a player chooses between two pure actions: C which stands for *cooperate* and D which stands for *defect*. The payoff function of any player $i \in I$ is defined as follows: for each $a \in A$,

$$h_i(a) = \begin{cases} 3 & \text{if } a_j = C, \ \forall\ j \in I \\ 0 & \text{if } a_i = C \text{ and } \exists\ j \in I \setminus \{i\} \text{ s.t. } a_j = D \\ 4 & \text{if } a_i = D \text{ and } a_j = C, \ \forall\ j \in I \setminus \{i\} \\ 2 & \text{if } a_i = D, \ \exists\ j \in I \setminus \{i\} \text{ s.t. } a_j = D \text{ and } \exists\ l \in I \setminus \{i, j\} \text{ s.t. } a_l = C \\ 1 & \text{if } a_j = D, \ \forall\ j \in I. \end{cases}$$

The unique Nash Equilibrium of stage game $\hat{G}$ is the action profile in which all players choose D, since it is a strictly dominant action. In the repeated Prisoner's Dilemma, however, it is possible to sustain strategy profiles that yield a higher payoff to all players and are SE under certain conditions, such as the trigger strategy profile. It prescribes each player to cooperate as long as all players cooperate and to defect forever if any player defected. Given any network g, the trigger strategy of player i, denoted by $\hat{f}_i \in F_i$, is defined as follows: $\hat{f}_i^1 = C$, and for $t \geq 1$, given $ob_i^t \in Ob_i^t$,

$$\hat{f}_i^{t+1}(ob_i^t) = \begin{cases} D & \text{if } \exists\ 1 \leq \tau \leq t \text{ such that for } a^\tau \in ob_i^t, \ a_j^\tau = D, \text{ while } a_{-j}^\tau = C \\ C & \text{otherwise.} \end{cases}$$

Given $\hat{f} \in F$, observe that for all $i \in I$ and all $t \geq 1$, first $a_i^t(\hat{f}) = C$, and second, $ob_i^t(\hat{f})$ is such that for all $a_j^\tau \in ob_i^t(\hat{f})$, $a_j^\tau = C$ as well for all $1 \leq \tau \leq t$ and all $j \in I$. Hence, for all $i \in I$, $H_i^\delta(\hat{f}) = (1-\delta)\sum_{t=1}^\infty \delta^{t-1} h_i(a^t(\hat{f})) = (1-\delta)\sum_{t=1}^\infty \delta^{t-1} 3 = 3$.

3.1 The Players form a Star

Consider a *star* (or a line) with $n = 3$, as represented in Figure 1. The graph of g is $E = ((1, 2), (2, 1), (2, 3), (3, 2))$. Figure 2 represents $\hat{G}$ for $n = 3$, where player 1 chooses rows, player 2 columns and player 3 matrices. The trigger strategy profile is a SE of $\hat{G}^{g,\delta}$ if, and only if, all players are patient enough, that is, δ is higher than some threshold value. Then, none of them ever deviates. Corresponding conditions on δ must be found

[14] The reader is referred to this paper for additional results and references.

1——2——3

Figure 1: Three players form a Star

3

C

1-2	C	D
C	3, 3, 3	0, 4, 0
D	4, 0, 0	2, 2, 0

D

1-2	C	D
C	0, 0, 4	0, 2, 2
D	2, 0, 2	1, 1, 1

Figure 2: Prisoner's Dilemma for three players

for the truncation of the repeated network Prisoner's Dilemma after any point in time, and therefore, given any observation profile. Three classes of unilateral deviations can be identified. A SE does not impose restrictions on play after a multilateral deviation by two or more players. Any unilateral deviation that may arise in the course of play can be uniquely allocated to one class. The three classes are

1) initial unilateral deviations,
2) subsequent unilateral deviations (before the initial is known by all players), and
3) unilateral deviations when the punishment takes place.

Obviously, unilateral deviations during the punishment are not profitable since all players play D. The resulting action profile is the stage game Nash Equilibrium in strictly dominant actions. Hence, every player best-replies independently of g and of δ. For the same reason, no player can deviate profitably from the trigger strategy profile in class 2. After a player's initial deviation, he and any player who knows about it are best-off to play D forever (rather than to deviate and to choose C at any point in time).

It remains to show that no player has a profitable unilateral deviation from the trigger strategy profile when all players play C. Given δ, player 2 (who is directly observed by 1 and 3) does not deviate in any period τ if, and only if,

$$(1-\delta)\sum_{t=1}^{\infty} 3\delta^{t-1} \;\geq\; (1-\delta)\sum_{t=1}^{\tau-1} 3\delta^{t-1} + 4(1-\delta)\delta^{\tau-1} + (1-\delta)\sum_{t=\tau+1}^{\infty} 1\delta^{t-1},$$

$$(1-\delta)\sum_{t=\tau+1}^{\infty} 2\delta^{t-1} \;\geq\; (1-\delta)\delta^{\tau-1},$$

$$2\delta^{\tau+1} \geq (1-\delta)\delta^{\tau},$$

$$\delta \geq \tfrac{1}{3}.$$

The value of $\frac{1}{3}$ is not only the threshold value for player 2 in this example but also the one for all players in a complete network. The network affects, however, the threshold value of the remaining two players in this example. Given δ, player 1 (and similarly 3) does not deviate from the trigger strategy profile in any period τ if, and only if,

$$(1-\delta)\sum_{t=1}^{\infty} 3\delta^{t-1} \geq (1-\delta)\sum_{t=1}^{\tau-1} 3\delta^{t-1} + 4(1-\delta)\delta^{\tau-1} + 2(1-\delta)\delta^{\tau} + (1-\delta)\sum_{t=\tau+2}^{\infty} 1\delta^{t-1},$$

$$(1-\delta)\delta^{\tau} + (1-\delta)\sum_{t=\tau+2}^{\infty} 2\delta^{t-1} \geq (1-\delta)\delta^{\tau-1},$$

which can be simplified to $2\delta+\delta^2-1 \geq 0$. The only positive solution for δ in this quadratic equation is approximately 0.414. Hence, in class 1 of the SE conditions the requirement on δ, or the players' patience, is higher in the star with three players considered here than in a complete network. This is due to the one period delay with which players 1 and 3 obtain information about each other's action choice.

This example extends to the case where $n > 3$ and the players form a star. The player at the center of the star has the same role as player 2 in this example, and for all other players the same conditions apply as for players 1 and 3 in this example.

3.2 The Repeated Prisoner's Dilemma Played in any Network

A similar result holds for any network (with $n > 3$). Suppose that all players in a network follow the trigger strategy profile. Then, analogously to the calculation for players 1 and 3 in the above example, a condition obtains such that no player $i \in I$ deviates. The corresponding expression is $2\delta + \delta^{d_i} - 1 \geq 0$. Although it depends on d_i, even in large networks the threshold value for δ is bounded above by $\frac{1}{2}$. Hence, for "moderately patient" players, the trigger strategy profile is a SE in any repeated network Prisoner's Dilemma.

The network may thus reduce the set of discount factors for which a strategy profile is a SE, and moreover, for a given discount factor, the set of SE strategy profiles and the corresponding set of payoff vectors may be strictly smaller in the repeated network game than in the version with complete network.[15] The next step is to extend this result to repeated network games based on any stage game.

[15] The reduction in the payoff space for $\delta \in [\frac{1}{3}, \frac{1}{2})$ is the point $(3,3,3)$, since the trigger strategy profile is no SE. See Kinateder (2008) for a formal proof and a graphical illustration of this result.

4 Information Sharing and Punishment Reward

In general, the conditions for SE are not as simple as in the repeated network Prisoner's Dilemma since the action profile forcing a player to his *effective minmax* payoff does not coincide with a stage game Nash Equilibrium in strictly dominant actions. Hence, punishment is asymmetric and may be costly for some players.

Until all players in the network know about an initial deviation, they follow the sequence of action profiles. Once all players have identified the initial deviator, they start to punish him. This phase of information transmission is called Information Sharing Process (ISP). Note, that the ISP-payoff is not normalized by $(1-\delta)$.

Definition 2. *Given $f \in F$, the* Information Sharing Process payoff *of player i following an initial deviation in period t' only is defined as*

$$ISP_i^{t'} \;=\; h_i(a^{t'+1}(f)) \;+\; ... \;+\; \delta^{d-2} h_i(a^{t'+d-1}(f)).$$

Note, that an action profile is known by all players after $d-1$ periods. The ISP can be extended easily to cover a deviation of finite length by any player. Any subsequent unilateral deviator with non-equivalent utility to the initial one starts a new ISP which may overlap with the ongoing one. Once every player has identified the last deviator, he is forced to his *effective minmax* payoff at least until his entire gain from deviating is taken away or until another subsequent deviator is punished. All players that contribute to the punishment may incur a loss in their own payoff as long as it lasts. Hence, punishment starts only once all players know about the deviation, it should be restricted to a minimal amount of time, and the punishers should be rewarded thereafter. Obviously, the reward must not be beneficial for the deviator—otherwise, the punishment is reversed again.

Assume without loss of generality that all NEU-players occupy positions $1, ..., \hat{i}$ in I, and that thereafter all players in the distinct EU-groups S_1 to S_U follow. In analogy to ADS, given any feasible and strictly individually rational target payoff vector $x \in \mathcal{F}^*$, for all NEU-players there are player-specific punishment reward payoff vectors denoted by $\omega^1, ..., \omega^{\hat{i}}$. They can be achieved by sequences of pure action profiles and have the following properties. For any player $i \notin S$, $x_i > \omega_i^i > 0$, and for two distinct players $i \nsim j$, $\omega_i^i < \omega_i^j$, that is, the i-th component of vector i is strictly smaller than that of any other one.

For EU-players the punishment reward phase is simpler. Some time after player $i \in S_u$ deviated, all members of S_u are subjected to the same punishment reward since their payoffs are equivalent to i's. Hence, for each group S_u, $1 \leq u \leq U$, there is one punishment reward payoff vector ω^{S_u}. A cascade of deviations by players in S_u is prevented by taking

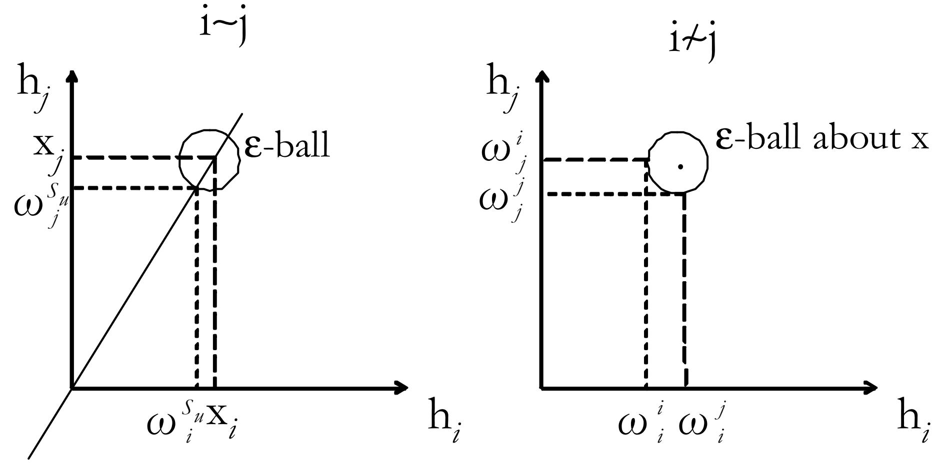

Figure 3: Construction of Punishment Reward Payoff Vectors for EU- and NEU-Players

away the gain each of them obtained, that is, by forcing the players in S_u to their *effective minmax* payoff long enough. Thereafter, the group's punishment reward phase is played.

Given any target payoff vector $x \in \mathcal{F}^*$, the punishment reward payoff vectors $\omega^1, ..., \omega^{\hat{i}}, \omega^{S_1}, ..., \omega^{S_U}$, have the following properties:

i) for all $i \notin S$, $x_i > \omega_i^i > 0$,
and for any $1 \leq u \leq U$ and all $i \in S_u$, $x_i > \omega_i^{S_u} > 0$.

ii) a) For all $i \neq j$, $i, j \notin S$, $\omega_i^i < \omega_i^j$,
b) for any $1 \leq u \leq U$, all $i \in S_u$ and all $j \notin S$, $\omega_i^{S_u} < \omega_i^j$ and $\omega_j^j < \omega_j^{S_u}$,
c) for all $i \in S_u$, $j \in S_{u'}$ such that $u \neq u'$, $\omega_i^{S_u} < \omega_i^{S_{u'}}$ and $\omega_j^{S_{u'}} < \omega_j^{S_u}$,
d) and for any $1 \leq u \leq U$, and all $i, j \in S_u$, there are $\alpha > 0$ and $\beta \in \mathbb{R}$ such that $\omega_i^{S_u} = \alpha \omega_j^{S_u} + \beta$.

The conditions in part i) are target payoff vector domination and individual rationality. The ones in part ii) ensure that a player is worst off during his or his EU-group's punishment reward phase, but that he can be rewarded otherwise.

The existence of the punishment reward payoff vectors for any $x \in \mathcal{F}^*$ follows from ADS, who construct them explicitly and give the following geometric interpretation, graphically illustrated in Figure 3. For two distinct players $i \nsim j$, the projection of the payoff space on the corresponding two player plane yields an ellipse or a line (with negative slope), whereas for all others it is a line (with positive slope). In the first case, the smallest i- and j-coordinates on a ball with arbitrarily small radius $\varepsilon > 0$ about the target payoff vector gives the payoff that the corresponding player receives in his punishment reward phase. In any other case, the EU-group's punishment reward payoff vector is the lowest point, in which the line and the ε-ball about the target payoff vector intersect. This intersection determines the punishment reward payoff of each player in the group.

5 The Results

5.1 Folk Theorem

Provided that the players are patient enough, a strategy profile can be constructed for which, given any observation profile, no player's unilateral deviation from the continuation strategy profile is profitable. Hence, this is a SE of the repeated network game and a Folk Theorem obtains. The outline and proof of the Folk Theorem (which can be found in Appendix A) adapt some arguments of ADS and Wen to the network case.

Theorem 1. *Let G and g be given. Then, for all $x \in \mathcal{F}^*$, there is $\tilde{\delta} < 1$ such that for each $\delta \in (\tilde{\delta}, 1)$, there is a corresponding $\tilde{f} \in F$ such that $\tilde{f} \in SE(G^{g,\delta})$ and $H^\delta(\tilde{f}) = x$.*

Strategy profile $\tilde{f}$ prescribes the players to punish a unilateral deviator once all of them know that he deviated at least until his entire gain is taken away. Thereafter, his or his EU-group's punishment reward phase is played. Since each observation profile that may arise can be uniquely allocated to one of a small number of classes of observation profiles, it is necessary and sufficient to show for each class that any player's finite unilateral deviation is not profitable. The Folk Theorem can be proved as well for other strategy profiles, as is discussed in section 5.3.

Patient players in the network do not mind to obtain the history of the repeated game gradually over time. Immediate punishment or punishment that sets in after a finite delay are equivalently strong threats for the players in this case. In the limit, the network effects disappear and the same set of payoff vectors can be generated by SE in the repeated game and in its network version.

Corollary 1. *Let G and g be given. Then, there is $\bar{\delta} < 1$ such that for all $\delta \in (\bar{\delta}, 1)$ and all $x \in \mathcal{F}^*$, there are $f \in SE(G^{g,\delta})$ and $\bar{f} \in SE(G^\delta)$ such that $\{a^t(f)\}_{t=1}^\infty \equiv \{a^t(\bar{f})\}_{t=1}^\infty$, and $H^\delta(f) = H^\delta(\bar{f}) = x$.*

Since network g is assumed to be undirected, a simple structure on the information transmission obtains. As already hinted in the introduction, however, the players may not be able to obtain information about each other simultaneously. The Folk Theorem extends to repeated games played on directed networks that are connected since each player still gets to know the repeated game's history with a finite delay. Apart from this, the observation and the communication network need not coincide. A player may observe a neighbor, though he may not be able to communicate with him. Denote by (I, E^{Ob}) and (I, E^{Com}) the *observation* and the *communication* graph of the observation network g^{Ob} and the communication network g^{Com}, respectively. The two graphs are defined as (I, E). However, both may be directed and fulfill the following connectedness property. Each player is observed by at least one other player. The players communicate their observations via a directed network g^{Com} such that all of them obtain the repeated network game's history after a finite delay. For any network g^{OC}, consisting of an observation network g^{Ob} and a communication network g^{Com}, the Folk Theorem holds.

Corollary 2. *Let G and g^{OC} be given. Then, for all $x \in \mathcal{F}^*$, there is $\dot{\delta} < 1$ such that for each $\delta \in (\dot{\delta}, 1)$, there is a corresponding $f \in F$ such that $f \in SE(G^{g^{OC},\delta})$ and $H^\delta(f) = x$.*

Corollary 2 shows that the results obtained in this paper are valid for a more general class of models which capture any finite and heterogeneous delay between information generation and the players' reaction to it.

Finally, note that for a given set of players the network in which the delay after which punishment starts is largest in a *tree*, that is, a line of length $n-1$. In this case, the diameter among all networks that can be formed from the set of players is maximal. Given G, let $\hat{g}$ be an arbitrary tree network formed by the players in set I.

Corollary 3. *Let G, $\hat{g}$ and $f \in F$ be given. Assume that $f \in SE(G^{\hat{g},\delta})$ for all $\delta \in (\hat{\delta}, 1)$. Then, for any g formed by set I and all $\delta \in (\hat{\delta}, 1)$, $f \in SE(G^{g,\delta})$ and $H^\delta(f) > 0$.*

In other networks than trees the diameter is lower, and hence also the requirement on the players' level of patience. In general, f may be a SE even for lower discount factors when the players form any other network than a tree.

5.2 Network Effects for Impatient Players

For impatient players, or in other words, for a range of discount factors strictly below 1, the network may make a difference. For the Prisoner's Dilemma this was shown in section 3. In this section, a similar result is derived for any stage game and any network.

A lower bound of the discount factor $\underline{\delta}$ can be identified such that for all $\delta \in [0, \underline{\delta}]$, only sequences of action profiles that prescribe the infinite repetition of stage game Nash Equilibria are supportable by SE in both games. Together with Corollary 1, the reduction in the set of sequences of action profiles that can be support by SE in a complete network but not given some other network g is then stated formally.

Corollary 4. *Let G and g be given. Then, there are $0 < \underline{\delta} \leq \bar{\delta} < 1$ such that for all $\delta \in (\underline{\delta}, \bar{\delta}]$, $\{\{a^t(f)\}_{t=1}^{\infty} \mid f \in SE(G^{g,\delta})\} \subset \{\{a^t(\bar{f})\}_{t=1}^{\infty} \mid \bar{f} \in SE(G^{\delta})\}$.*

For a range of intermediate discount factors, the network reduces the set of sequences of action profiles that are generated by SE strategy profiles. When the network is complete or in very special other cases, the lower and upper bound of δ coincide and the corollary is trivially true. It is taken into account that other strategy profiles than $\tilde{f}$ may yield the Folk Theorem for discount factors below $\tilde{\delta}$, identified in Theorem 1 (see section 5.3).

Finally, formal conditions are identified under which the network reduces the set of SE strategy profiles for impatient players. Given G, δ and g, assume that $\bar{f} \in SE(G^{\delta})$ and let $\{\dot{a}^t\}_{t=1}^{\infty} \equiv \{a^t(\bar{f})\}_{t=1}^{\infty}$. Say that *the network has an impact with respect to* $\tilde{f}$, as defined in Theorem 1, if $\tilde{f}$ does not support $\{\dot{a}^t\}_{t=1}^{\infty}$ as a SE of $G^{g,\delta}$. (Note, however, that this does not rule out that there is some other strategy profile $f \neq \tilde{f}$ such that $f \in SE(G^{g,\delta})$ and $\{a^t(f)\}_{t=1}^{\infty} = \{\dot{a}^t\}_{t=1}^{\infty}$.) Suppose that player i can gain

$$\beta_i^{\tau} \equiv \sum_{t=\tau}^{\tau+d-1} \delta^{t-\tau}[\max_{a_i \in A_i} h_i(a_i, \dot{a}_{-i}^t) - h_i(\dot{a}^t)]$$

by a deviation of length $d-1$ from $\{\dot{a}^t\}_{t=1}^{\infty}$ that starts at τ. If $i \notin S$, let

$$\lambda_i^{\tau}(T) \equiv \sum_{t=\tau+d}^{\infty} \delta^{t-\tau-1} h_i(\dot{a}^t) - (1-\delta)^{-1}\delta^T \omega_i^i$$

for $T \geq 2d-2$. It takes $d-1$ periods until all players know about i's deviation, and $2d-2$ periods after it, all of them know if i deviated again one period before his punishment started. An analogous expression can be obtained when $i \in S$. Then, Proposition 1 identifies conditions under which *the network has an impact with respect to* $\tilde{f}$.

Proposition 1. *Let G, $\delta < 1$ and g be given. Suppose there is $\bar{f} \in SE(G^{\delta})$, $i \in I$ and $\tau \geq 1$, such that for all positive integers $T \geq 2d-2$, $\beta_i^{\tau} > \lambda_i^{\tau}(T)$. Then, the network has an impact with respect to $\tilde{f}$.*

Appendix B contains the proof of Proposition 1. Intuitively, player i deviates from $\{\dot{a}^t\}_{t=1}^{\infty}$, if the punishment threat of strategy profile $\tilde{f}$ is discounted by too much, and hence, not strong enough to prevent i's deviation. Thus, the strategy profile defined in Theorem 1 does not support the sequence of action profiles $\{\dot{a}^t\}_{t=1}^{\infty}$ as a SE of $G^{g,\delta}$, and *the network has an impact with respect to* $\tilde{f}$. Similar conditions can be identified for any other strategy profile than $\tilde{f}$. In case *the network has an impact with respect to* all $f \in F$, given δ, then the sequence of action profiles $\{\dot{a}^t\}_{t=1}^{\infty}$ cannot be generated by a SE strategy profile in the repeated network game $G^{g,\delta}$. If this holds for all $\bar{f} \in SE(G^{\delta})$ which generate one payoff vector, then this payoff vector cannot be generated by a SE in the repeated network game. Hence, a reduction in the set of equilibrium payoff vectors obtains. The condition for this is expressed formally in Corollary 5.

Corollary 5. *Let G, $\delta < 1$ and g be given. Suppose for all $\bar{f} \in SE(G^{\delta})$ with $H^{\delta}(\bar{f}) = x$, the network has an impact with respect to all $f \in F$. Then for all $f \in SE(G^{g,\delta})$, $H^{\delta}(f) \neq x$.*

This result shows under which conditions the network reduces the set of equilibrium payoff vectors by one payoff vector $x \in \mathcal{F}^{*}$. To characterize the entire set of equilibrium payoff vectors for a given discount factor and network compared with a complete one, would yield the network's overall effect on the set of equilibrium payoff vectors. However, this task is non-trivial and only Cornshaw (1997), for a complete network, fully characterized the payoff space of a three player (Cournot) game as a function of the discount factor. The interested reader is referred to Kinateder (2008) who illustrates analytically and graphically the reduction in the payoff space for a star versus a complete network each formed by three players which play the repeated Prisoner's Dilemma game.

The interplay between the delay in information transmission caused by the network and the patience of a player reduces the set of SE and the corresponding payoff set under certain conditions, although for large enough discount factors the Folk Theorem holds. Comparative statics on g have similar effects on $SE(G^{g,\delta})$, when due to the removal or addition of a link the network's diameter or some player's largest distance changes.

5.3 Other Strategy Profiles

For other strategy profiles than $\tilde{f}$, the Folk Theorem may hold even for smaller discount factors than $\tilde{\delta}$, identified in Theorem 1. Two strategy profiles which achieve this are described. Under both, the players use the information they receive earlier than under $\tilde{f}$.

Given any network, a player can start to punish a deviator, for example, when he knows the action profile of the period, in which the deviation occurred. Hence, with respect to any player i, the time delay, with which the players identify i's unilateral deviation, induces a partition of the set of players such that all players in a group observe the action profile played in the period of i's deviation with the same delay. After some initial delay, during which i's deviation is unpunished, the players which first identify i's deviation start to punish him. Since the network is connected the group of punishers, thereafter, grows strictly in each period until it comprises all players, d periods after i's deviation. At the end of the previous period, all players know about the deviation.

Alternatively, a player may immediately punish any neighbor's deviation.[16] In subsequent periods the group of punishers grows strictly until it comprises all players. The delay until this is the case is determined by the deviator's largest distance. For at least one pair of players, this coincides with the network's diameter (which is the maximal largest distance between any pair of players).

To illustrate both ideas, consider a network whose graph is as depicted in Figure 1 and any stage game G. A unilateral deviation by player 1 (and similarly by player 3) is immediately identified by player 2 since he also observes player 3's action choice in the period of player 1's deviation. Hence, from the subsequent period on, player 2 punishes player 1. Player 3 contributes to the punishment only from one period afterwards on. This version of the trigger strategy profile was used in section 3 to show that the network makes a difference in the Prisoner's Dilemma.

In the repeated network game, the diameter of the network thus determines when the group of punishers comprises all players. Only then, punishment is as effective as in a complete network already one period after the deviation. Hence, the threat of punishment in any network is always equally or less strong than in a complete one. However, which strategy profile supports a sequence of action profiles as a SE for a lower discount factor depends on the network, the stage game and the prescribed sequence of action profiles.

5.4 Network Analysis

The result that in a repeated game played on a fixed network its diameter determines whether cooperation is sustainable, for a given discount factor, is novel. Conversely, the network literature emphasizes that cooperation depends on the *clustering coefficient*, which gives the ratio of triads or circles of three players in a network relative to all possible combinations of three players in I. Whereas the diameter is a global measure, the clustering coefficient measures local connectedness. Its importance in the network literature is due to two sociology papers. Granovetter (1973) defines the concept of *strong links* which exist, for example, between three friends when they form a triad (or a circle). This facilitates cooperation since the three friends mutually observe each other's behavior and since a person's friend's friend is his friend. Coleman (1988), in turn, develops the concept of *closures*, which are circles of connected people of any size (and not only three).

To see that a lower diameter in a network need not imply a higher clustering coefficient,

[16] Suppose $S = \emptyset$, that i deviates at t' from $\{a^t\}_{t=1}^{\infty}$ and that any deviator is "minmaxed" immediately. Then, at $t'+1$, i's payoff is $\min_{a_{i(1)} \in \times_{j \in i(1)} A_j} \max_{a_i \in A_i} h_i(a_i, a_{i(1)}, a^{t'+1}_{-(i \cup i(1))}) \leq \max_{a_i \in A_i} h_i(a_i, a^{t'+1}_{-i})$. In case $S \neq \emptyset$, an analogous condition can be found.

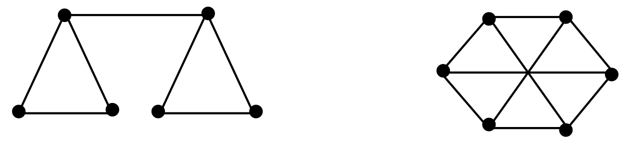

Figure 4: a) two triads b) no triad

consider the two networks depicted in Figure 4; for both $n = 6$. The *scales* in part a) has 7 links and two triads, whereas the *wheel* in part b) has 9 links and no triad. The diameter of network a) is 3 and the one in part b) is 2. The clustering coefficient of the wheel is zero, whereas the scales' one is positive. Hence, the relationship between the clustering coefficient and the diameter in a network need not be monotonic. (Obviously, other examples could be constructed in which the monotonic relation holds.)

Cooperation in the setup of this paper can be sustained more easily in network b). Given any stage game, for a certain range of discount factors, there are SE in network b) which generate sequences of action profiles that do not arise from SE in network a). (However, as the Folk Theorem implies, for patient players this difference disappears.) Conversely, in the network literature, network a) would fare much better in terms of sustaining cooperation than the one in part b) of Figure 4.

The clustering coefficient is important, for example, in Lippert and Spagnolo (2005) and in Vega-Redondo, Marsili, and Slanina (2005). Both let each linked pair of players play a bilateral repeated discounted Prisoner's Dilemma. In the first paper, a player's deviation severs the corresponding link. Closures are crucial to sustain cooperation, modelled in form of SE, since the information about a deviation is communicated through the network. In the second paper, payoffs are stochastically decaying over time. A player severs a link once his payoff falls below some threshold, although he is then punished by all mutual neighbors the two players have. A higher clustering coefficient again fosters cooperation.

Haag and Lagunoff (2006) let each neighborhood play a repeated Prisoner's Dilemma. They study optimal network design when each player's discount factor is randomly determined before the game begins and do not find that the diameter matters.

Since usually a player's payoff depends not only on his and his neighbors' decisions but also on decisions of other players in the network, even if they are "far away", it seems realistic to consider repeated games played on a fixed network.[17] In other closely

[17] Though a network formation game might precede the repeated network game, this setup can be easily extended to explicitly take into account incentives to form or maintain links since comparative statics on

related papers, however, closures are also decisive to sustain equilibria. In Ben-Porath and Kahneman (1996) a SE obtains in a repeated game with public announcements if there are at least three players in each group. Then, any lie is detectable in equilibrium. This is exactly identical to strong links. In their paper with costly monitoring, Ben-Porath and Kahneman (2003) require a similar condition to hold. Every player is assigned one monitor. After an incompatible announcement both are punished and the monitor is substituted. Again, three players are necessary to prevent profitable unilateral deviations. In Renault and Tomala (1998), in which strategic communication includes lies as well, cooperation is sustainable only in 2-connected networks. Then, two distinct paths exist between any pair of players and this is just the formal description of a closure.

The delayed perfect monitoring model yields a different result since it assumes bilateral communication. Players become informed about the repeated network game's history gradually over time. To the contrary, after a public announcement in Ben-Porath and Kahneman (1996 and 2003), all players immediately punish the deviator. In models, in which communication, observations and the repeated games played are bilateral, punishment is also immediate. In Renault and Tomala (1998), play is interrupted until all players know who has cheated. Simultaneously, the payoff accumulation stops, which is unimportant since the payoff is undiscounted. In the repeated network game, the impatient players—except of the deviator—suffer from the delay, during which punishment is less effective than in a complete network. The diameter of the network captures this delay and determines together with the discount factor whether a strategy profile is a SE. This result extends to both strategy profiles mentioned in section 5.3. Even if punishment started earlier than under $\tilde{f}$, defined in Theorem 1, clustering is harmful since redundant information is spread among the same players. Again the network's diameter is decisive since it determines once all players punish a deviator. Other assumptions may also contribute to the different outcomes. In particular, a deeper analysis of the matrices that contain the distinct networks (see footnote 1) might yield interesting results.

6 Final Remarks

6.1 Mixed Actions

The extension of the Folk Theorem to mixed actions is straightforward in the complete network. Additionally, a player's deviation within the support of his mixed action, which

the topology of the network are straightforward, as mentioned in section 5.2.

is not observed by the other players, must be prevented. FLT achieve this in the complete network setting by making future play dependent on the realized action profile today. By letting a high payoff today follow a low one tomorrow, and vice versa, FLT make each player exactly indifferent among all pure actions in the support of a mixed one.[18]

The Folk Theorem for the repeated network game can be extended to mixed actions using FLT's idea. A deviator is forced to his *effective minmax* payoff in mixed actions—apart from punishment, mixed actions are not used. After the number of periods equivalent to the diameter of the network passed, every player knows the pure action profile generated by the mixed action in the first punishment period. Punishment continues, anyway, at least until this period, and then, FLT's strategy is used to compensate the players for their choices in the first punishment period. Thereafter, the second punishment period is compensated, and so on. This process stops in finite time. The extension of this model to mixed actions would allow a larger set of payoff vectors to be sustained by SE. However, patient players can achieve first best outcomes already with pure actions.

6.2 Strategic Communication

Strategic Communication can be modelled either by letting the players decide whether to transmit information or not, or by allowing them to lie. A player can be easily prevented from stopping the information transmission by the threat to punish him as if he had deviated. The second type of deviation, therefore, is more interesting but also more involved. In a SE initial as well as subsequent deviations have to be prevented and a sequence of different liars and deviators may be difficult to disentangle for a player. However, a lie may be part of a SE and new equilibria emerge due to richer communication.

That players may lie is assumed, for example, in Compte (1998) and Kandori and Matsushima (1998). They model imperfect private monitoring in repeated games (without network) as follows. Each player receives a distinct distorted private signal of the period's action profile. By publicly announcing these private observations every $K > 0$ periods, the players restore a public history on which they condition their action choices and a Folk Theorem obtains. Without communication the players' beliefs about where in the game tree they are might diverge and some player's profitable deviation may be undetectable. Kandori (2003) uses a similar idea in the case of imperfect public monitoring in which all players observe the same imperfect signal of the period's action profile

[18] The main difficulty arises for players with positively and negatively related payoff functions since their payoff space is a line. Since it is only possible to move "up" and "down" the line in the future, it is non-trivial to define a continuation strategy which makes each player indifferent among the support of the mixed action. However, FLT construct a strategy profile which achieves this task in finite time.

and publicly announce their own action choices. Under strategic communication a Folk Theorem obtains under weaker conditions than without communication.[19] In all cases, a payoff transfer mechanism provides incentives for the players to make truthful announcements. A player's payoff increases or decreases depending on his announcement. A similar mechanism induces the players in Ben-Porath and Kahneman (1996 and 2003) to truthfully announce their own and any neighbor's action choice publicly. These constructions require a full-dimensional payoff space which is even stronger than the NEU-condition.

In the repeated network game, the payoff of an EU-group may have to be increased and decreased at the same time under such a payoff transfer mechanism. This, however, is impossible. The presence of EU-players creates another problem. Suppose that two or three players that monitor each other belong to the same EU-group. Then, cooperation immediately breaks down because all other players anticipate a sequence of deviations by the EU-players which these, obviously, do not reveal when communicating their mutual observations of each other. Since in this setup a player does not communicate his own action choice, but only those of his neighbor(s), the problem can be solved by isolating the EU-players. They would, for example, occupy the positions of players 1 and 3 in the graph depicted in Figure 1, while a NEU-player would take player 2's.[20]

Nevertheless, due to the bilateral communication structure, it is challenging to introduce strategic communication to the repeated network game. Each player receives different information gradually over time, and hence, all players never simultaneously condition their action choices on the same (communicated) information. To prevent lies, therefore, requires, apart from isolating the EU-players in a network, to adapt the punishment reward phase. In order not to tell the lie that another player deviated, for each player $i \in I$, let $x_i > \omega_i^j$ for all $j \notin S$, and $x_i > \omega_i^{S_u}$ for all S_u. By lying a player makes himself worse off under this condition. To induce a player to truthfully reveal any neighbor's deviation, additionally, for all $i \in I$, let $\omega_i^k > \max_{a_k \in A_k} h_i(a_k, a_{-k}^t)$ for all $k \in i(1)$ with $k \notin S$ and any $t \geq 1$. An analogous condition must hold when $k \in S$. Then, by tolerating k's deviation i is worse off than by reporting it. The information that k deviated flows throughout the network only, if this incentive constraint holds sequentially for all players in $k(1)$, $k(2)$, and so on. Then, lies are unprofitable for δ close enough to 1 and restricting the players to initial deviations only. After a history that includes lies and deviations, the construction of the punishment reward payoff vectors may have to be revised in order to maintain the incentives for truthtelling and complying with the strategy profile.

[19] The Folk Theorem under imperfect public monitoring without communication in Fudenberg, Levine and Maskin (1994) holds if the public signal allows the players to statistically detect unilateral deviations.

[20] Cooperation can be sustained in any star as well when the player at the center has a constant payoff.

For the Prisoner's Dilemma, as defined in section 3, Kinateder (2008) shows that, given any network, the players that follow the trigger strategy profile tell the truth even if they could lie as long as $\delta \geq \frac{1}{2}$, the threshold value identified in section 3.2. Many other strategy profiles are not robust to strategic communication and truthtelling does not obtain as a SE since at least one player has a profitable lie. The set of SE strategy profiles under exogenously imposed truthtelling intersects with that under strategic communication although none is a subset of the other. To extend the repeated network game based on any stage game to strategic communication is left for future research.

6.3 Conclusion

In this paper, delayed perfect monitoring in an infinitely repeated discounted game is modelled by allocating the players to a connected (and undirected) network. Under truthtelling, the Folk Theorem obtains since patient players do not mind to receive the repeated game's history gradually over time. For impatient players the network makes a difference which need not be big, as shown for the Prisoner's Dilemma. The interplay between the diameter of the network and the patience of the players leads to the reduction in the set of SE for a given discount factor. Due to the network, the set of equilibrium payoff vectors is reduced for a range of discount factors and a strategy profile is a SE over a smaller range of discount factors, both compared with a complete network. This paper also contributes to the network literature by emphasizing the importance of the diameter, and not the clustering coefficient, to sustain cooperation in a network.

This setup applies in various other contexts. Not only companies, but also impatient people form networks and interact strategically over time, such as within a company, in any other organization, or in society at large. As long as all of them are on the same hierarchical level, this model applies. Also macroeconomic applications can be thought of. The players in a network thus might be all the companies in an economy and a deviation could be interpreted as one of them going bankrupt.

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Appendix A Proof of Theorem 1

Given G and g, fix $x \in \mathcal{F}^*$ such that x is feasible (see footnote 12). Strategy profile $\tilde{f} \in F$, which after being defined is shown to be a SE of $G^{g,\delta}$ for any $\delta \in (\tilde{\delta}, 1)$, prescribes a different sequence of pure action profiles $\{a^s\}_{s=1}^{\infty}$ to yield x for each δ, although its structure is unchanged. For any $j \in I$, define $\tilde{f}_j \in F_j$ as follows:
$\tilde{f}_j^1 = a_j^1$, and for $t > 1$, given $ob_j^{t-1} \in Ob_j^{t-1}$, in a slight abuse of notation, let $\tilde{f}_j^t(ob_j^{t-1}) =$

1) a_j^t, unless there is $1 \leq t' < t$ such that for $\hat{a}^{t'} \in ob_j^{t-1}$, $\hat{a}_i^{t'} \neq a_i^{t'}$, while $\hat{a}_{-i}^{t'} = a_{-i}^{t'}$. In this case, switch to phase 2 at $t' + d_j$ and let $\tilde{a}_j^s = a_j^s$, for all $s \geq 1$.

2) $\tilde{a}_j^t$, if $t' + d_j \leq t < t' + d$, unless player l, where $l \neq i$ and $l \notin S_u$ if $i \in S_u$, deviates at any t'', where $t' < t'' < t' + d$. Then, restart phase 2, set $t' = t''$ and choose $\tilde{a}_j^s$ accordingly. Otherwise, switch to phase 3 at $t' + d$.

3) $\bar{a}_j^i$, if $t' + d \leq t \leq t' + T$, where T is determined below. If any player l deviates at any $\bar{t}$, where $t' + T \geq \bar{t} \geq t' + d$, restart phase 2, set $t' = \bar{t}$ and choose $\tilde{a}_j^s$ accordingly. Otherwise, switch to phase 4 at $t' + T + 1$.

4) c_j^s, if $t \geq t' + T + s$, where $\{c^s\}_{s=1}^{\infty}$ is the sequence of action profiles that yields either ω^i if $i \notin S$, or ω^{S_u} if $i \in S_u$. If any player l deviates at any $\tau > t' + T$, restart phase 2, set $t' = \tau$ and choose $\tilde{a}_j^s$ accordingly. If $l = i$ or $i, l \in S_u$, restart $\{c^s\}_{s=1}^{\infty}$ where it was truncated by l's deviation, once phase 4 is reached again.

Phase 2 corresponds to the ISP, phase 3 to the *effective minmax* punishment of the last deviator, and phase 4 to the punishment reward phase. After any subsequent unilateral deviation, the phase in which the game is at the time of the deviation prescribes the play of the following $d-1$ periods—in general, phase 2 is restarted. Then, the new deviator is punished. In case, the same player deviates again in phase 2 (and no other one does), however, phase 2 is not restarted, but his punishment begins d periods after his first deviation. His entire gain is eliminated by forcing him to his *effective minmax* payoff for at least $d-1$ periods. After $d-1$ punishment periods, all players know if he deviated again in the period before it started, and hence, for how long it has to last in order to eliminate his entire gain. A similar argument applies for several unilateral deviations by distinct players of an EU-group during phase 2. After punishing the initial deviator for at least $d-1$ periods, the gain of the subsequent one(s) is eliminated.

By construction, the players can ignore multilateral deviations from $\tilde{f}$. It remains to show that no player's unilateral deviation from $\tilde{f}$ is ever profitable for large enough δ. The Folk Theorem holds trivially when a^t is a stage game Nash Equilibrium for all t, and only strategy profiles that do not generate such sequences of action profiles are considered.

The proof is organized as follows. The result for phase 2 is shown first since it introduces arguments used thereafter to prove the results of phases 4, 1 and 3. Note, that the following 6 combinations of players' deviations have to be shown to be unprofitable; for the first four $i \nsim j$ holds, whereas for the remaining two $i \sim j$ holds: $i \neq j$ and either $i, j \notin S$; or $i \in S$, but $j \notin S$; or $j \in S$, but $i \notin S$; or $i \in S_u$, $j \in S_{u'}$ such that $u \neq u'$; and finally, $i, j \in S_u$, or $i = j$. For each phase, the proof proceeds in this order. However, the last two cases are equivalent since $i \sim j$. This holds for $i, j \in S_u$ and if $i \notin S$, then $i = j$. Hence, only one case is proved explicitly for each phase since it implies the other.

PHASE 2

Figure 5 illustrates the order of time periods in phase 2. Suppose player $i \notin S$ deviated at t'. During the ISP player $j \neq i$, $j \notin S$, receives $ISP_j^{t'}$. By deviating at t'', where $t' < t'' < t' + d$, he can maximally gain $b_j = \max_{a \in A}[\max_{a_j \in A_j} h_j(a_j, a_{-j}) - h_j(a)]$, since his remaining ISP-payoff is unchanged. However, from period $t''+d$ on, he is forced to his *effective minmax* payoff of 0, and then, his punishment reward phase is played. Player j's deviation at t'' is not profitable when for some positive integer $\hat{T}_2$, where $t''+d \leq t' + \hat{T}_2$,

$$(1-\delta)b_j + \delta^{\hat{T}_2}\omega_j^j - (1-\delta)\sum_{t=t''+d}^{t'+\hat{T}_2} \delta^{t-t''-1} h_j(\bar{a}^i) - \delta^{t'+\hat{T}_2-t''}\omega_j^i \quad < \quad 0,$$

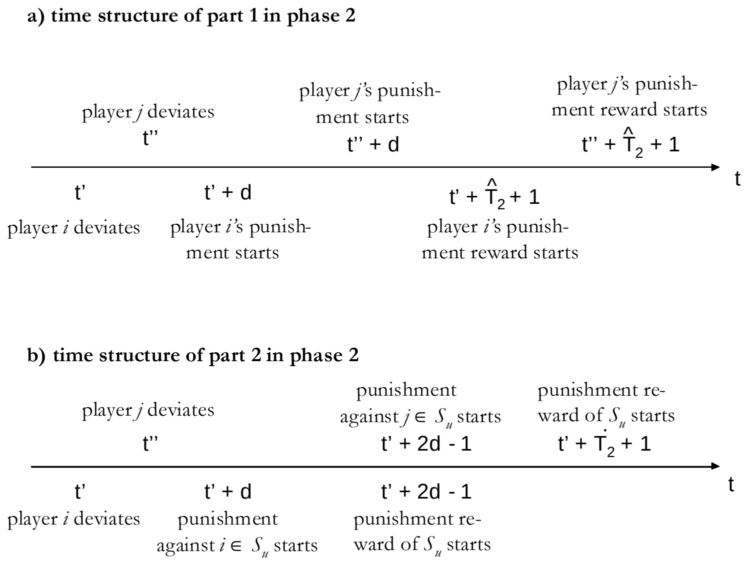

Figure 5: Order of time periods in phase 2

$$(1-\delta)b_j - (1-\delta)\sum_{t=t''+d}^{t'+\hat{T}_2} \delta^{t-t''-1} h_j(\bar{a}^i) \quad < \quad \delta^{t'+\hat{T}_2-t''}\omega_j^i - \delta^{\hat{T}_2}\omega_j^j. \tag{4}$$

Substituting $\delta^{t'+\hat{T}_2-t''}$ with $\delta^{\hat{T}_2}$ makes the right-hand-side of (4) smaller. (Since $t'' > t'$, $\delta^{t'+\hat{T}_2-t''} > \delta^{\hat{T}_2}$ holds for all $\delta < 1$.) Hence, (5) implies (4) and it suffices to show (5).

$$(1-\delta)b_j - (1-\delta)\sum_{t=t''+d}^{t'+\hat{T}_2} \delta^{t-t''-1} h_j(\bar{a}^i) \quad < \quad \delta^{\hat{T}_2}[\omega_j^i - \omega_j^j] \tag{5}$$

As δ converges to 1, (5) is fulfilled: its left-hand-side converges to zero whereas its right-hand-side is strictly positive since $\omega_j^i > \omega_j^j$. This may hold for several distinct pairs of discount factor and strictly positive integer. (The last inequality is fulfilled trivially when player j's gain from punishing player i is larger than b_j.) An analogous argument holds whenever $i \nsim j$. The case $t'' + d > t' + \hat{T}_2$ is simpler since the sum on the left-hand-side of (5) and j's payoff in the first period(s) of i's punishment reward phase both drop out, which for δ close to 1 is negligible.

For $i, j \in S_u$, after player j's deviation at any t'', where $t' < t'' < t' + d$, the ISP about i's deviation continues. Once all players know about i's deviation, $\bar{a}^i$ is played for at least $d-1$ periods, that is, at least until period $t'+2d-2$. Then, $\bar{a}^j \equiv \bar{a}^i$ is played until period $t'+\dot{T}_2$ to take away player j's gain from deviating at t''. Since j's punishment lasts at least one period, $\dot{T}_2 > 2d-2$. Thereafter, the EU-group's punishment reward phase is played. Player j's deviation at t'' is not profitable, if for some positive integer $\dot{T}_2 > 2d-2$,

$$
\begin{aligned}
(1-\delta)b_j + \delta^{t'+\dot{T}_2-t''}\omega_j^{S_u} - \delta^{t'+2d-2-t''}\omega_j^{S_u} &< 0, \\
(1-\delta)b_j &< (\delta^{t'+2d-2-t''} - \delta^{t'+\dot{T}_2-t''})\omega_j^{S_u}, \\
b_j &< \delta^{t'+2d-2-t''}(1-\delta)^{-1}(1-\delta^{\dot{T}_2-(2d-2)})\omega_j^{S_u}.
\end{aligned}
$$

When δ converges to 1, the right-hand-side converges to $(\dot{T}_2 - 2d + 2)\omega_j^{S_u} > 0$, by l'Hospital. Since b_j is a fixed positive number, the inequality is fulfilled for a large enough $\dot{T}_2$. By the same argument, no series of players from S_u or the same player, that is, $i = j$, can deviate profitably during the ISP about i's deviation. Finally, select a large enough, strictly positive integer T_2 such that no player can deviate profitably in phase 2.

PHASE 4 and PHASE 1

The result for phase 4 is stated first since it implies the result for phase 1. Suppose that player $i \neq j$, that $i, j \notin S$, and that i is the last deviator. Player j does not deviate at τ, the first period of i's punishment reward phase, if for some positive integer $\hat{T}_4$,

$$
\begin{aligned}
(1-\delta)\max_{a_j \in A_j} h_j(a_j, c^1_{-j}) + \delta(1-\delta)ISP_j^\tau + \delta^{\hat{T}_4}\omega_j^j - \omega_j^i &< 0, \\
(1-\delta)\max_{a_j \in A_j} h_j(a_j, c^1_{-j}) + \delta(1-\delta)ISP_j^\tau &< \omega_j^i - \delta^{\hat{T}_4}\omega_j^j.
\end{aligned}
$$

When δ converges to 1, the left-hand-side of the last inequality converges to zero whereas the right-hand-side is strictly positive (since $\omega_j^i > \omega_j^j$, and for any $\delta < 1$, $\delta^{\hat{T}_4} < 1$). The same argument holds whenever $i \nsim j$, and when player j deviates in any other than the first period of player i's punishment reward phase since for δ close to 1, the payoff obtained at the beginning of any punishment reward phase is negligible.

If $i = j$, player i cannot deviate profitably in the $\hat{\tau}$th period of his own punishment reward phase, if there is a positive integer $\dot{T}_4$ such that

$$
(1-\delta)b_i + \delta(1-\delta)ISP_i^\tau + \delta^{\dot{T}_4}\omega_i^i|_{s=\hat{\tau}+1}^\infty - \omega_i^i|_{s=\hat{\tau}+1}^\infty < 0,
$$

where $\tau \equiv t' + \dot{T}_4 + \hat{\tau}$ and $\omega_i^i|_{s=\hat{\tau}+1}^\infty \equiv (1-\delta)\sum_{s=\hat{\tau}+1}^\infty \delta^{s-1}h_i(c^s)$. This simplifies to

$$
(1-\delta)b_i + \delta(1-\delta)ISP_i^\tau < \omega_i^i|_{s=\hat{\tau}+1}^\infty - \delta^{\dot{T}_4}\omega_i^i|_{s=\hat{\tau}+1}^\infty \ ,
$$

$$b_i + \delta ISP_i^{\tau} \quad < \quad \tfrac{(1-\delta^{\dot{T}_4})}{(1-\delta)} \omega_i^i|_{s=\hat{\tau}+1}^{\infty} \ . \tag{6}$$

When δ converges to 1, the left-hand-side of (6) is bounded above by a positive number and the right-hand-side, by l'Hospital, converges to $\dot{T}_4 \omega_i^i|_{s=\hat{\tau}+1}^{\infty} > 0$. (Although, $\omega_i^i|_{s=\hat{\tau}+1}^{\infty}$ differs from ω_i^i, for δ close to 1, this difference is negligible and $\omega_i^i|_{s=\hat{\tau}+1}^{\infty}$ has the same properties as ω_i^i.) For $\dot{T}_4$ large enough, (6) holds. A similar argument applies when $i, j \in S_u$, and j deviates in the punishment reward phase of his EU-group. This argument together with the one used in phase 2 above demonstrate that no player's unilateral deviation of finite length is profitable in phase 4. Finally, let T_4 be the smallest positive integer such that no player can deviate profitably in phase 4.

The result of phase 4 extends to phase 1 since by assumption any player's target payoff is strictly larger than his punishment reward payoff. Moreover, neither finite deviations by one player nor subsequent deviations by distinct players in an EU-group are profitable in phase 1. Hence, also for phase 1 there is a discount factor $\delta < 1$ and a positive integers T_1 such that no player can deviate profitably from strategy profile $\tilde{f}$.

PHASE 3

Suppose player i is forced to his *effective minmax* payoff because he deviated at t'. By definition, neither player i nor any player $j \sim i$ can deviate profitably in this phase. Hence, suppose $i, j \notin S$. Player j does not deviate at any $\bar{t}$, where $t' + d \leq \bar{t} \leq t' + T_3$, if

$$(1-\delta)b_j + \delta(1-\delta)ISP_j^{\bar{t}} + \delta^{T_3}\omega_j^j - (1-\delta)\sum_{t=\bar{t}}^{T_3} \delta^{t-\bar{t}} h_j(\bar{a}^i) - \delta^{t'+T_3-\bar{t}}\omega_j^i \quad < \quad 0,$$

$$(1-\delta)b_j + \delta(1-\delta)ISP_j^{\bar{t}} - (1-\delta)\sum_{t=\bar{t}}^{T_3} \delta^{t-\bar{t}} h_j(\bar{a}^i) \quad < \quad \delta^{t'+T_3-\bar{t}}\omega_j^i - \delta^{T_3}\omega_j^j. \tag{7}$$

Proceeding as in phase 2, that is, substituting on (7)'s right-hand-side $\delta^{t'+T_3-\bar{t}}$ with δ^{T_3} (for any $\delta < 1$, $\delta^{T_3-(\bar{t}-t')} > \delta^{T_3}$ since $\bar{t} > t'$) and taking the limit of δ converging to 1, fulfills (7) for at least one pair of discount factor $\delta < 1$ and strictly positive integer T_3. An analogous argument holds for deviations, or a sequence of deviations, by EU- and NEU-players. Choose T_3 large enough to prevent any such deviation.

Let $T = \max\{T_1, T_2, T_3, T_4\}$, and let $\tilde{\delta}$ be the lowest discount factor, for which, given T, no player can deviate profitably in any phase. (If there are several pairs of T

and δ for which the proof holds, the pair with the lowest discount factor is selected.) Then, for any $\delta \in (\tilde{\delta}, 1)$, $\tilde{f}$ is a SE strategy profile of $G^{g,\delta}$ and $H^{\delta}(\tilde{f}) = x$.

Appendix B Proof of Proposition 1

Fix G, $\delta < 1$ and g. Select $\bar{f} \in SE(G^{\delta})$ that generates the sequence of action profiles $\{a^t(\bar{f})\}_{t=1}^{\infty} \equiv \{\dot{a}^t\}_{t=1}^{\infty}$. Take a strategy profile with the same structure as $\tilde{f}$, defined in Theorem 1, to support this sequence of action profiles as a SE of $G^{g,\delta}$. Then, *the network has an impact with respect to* $\tilde{f}$ if some player can deviate profitably. Consider first, that for some $i \notin S$, some $\tau \geq 1$, and all positive integers $T \geq 2d-2$,

$$(1-\delta)\sum_{t=\tau}^{\tau+d-1} \delta^{t-\tau} \max\nolimits_{a_i \in A_i} h_i(a_i, \dot{a}_{-i}^t) + \delta^T \omega_i^i \quad > \quad (1-\delta)\sum_{t=\tau}^{\infty} \delta^{t-\tau} h_i(\dot{a}^t),$$

$$\sum_{t=\tau}^{\tau+d-1} \delta^{t-\tau}[\max\nolimits_{a_i \in A_i} h_i(a_i, \dot{a}_{-i}^t) - h_i(\dot{a}^t)] + (1-\delta)^{-1}\delta^T \omega_i^i \quad > \quad \sum_{t=\tau+d}^{\infty} \delta^{t-\tau-1} h_i(\dot{a}^t).$$

Subtracting $(1-\delta)^{-1}\delta^T \omega_i^i$ from both sides yields $\beta_i^{\tau} > \lambda_i^{\tau}(T)$. *The network has an impact with respect to* $\tilde{f}$ if either the last inequality holds for some $i \notin S$ or an analogous condition for some $i \in S$. In the second case, ω_i^i is substituted with $\omega_i^{S_u}$.